The Direction of Technological Change on Renewable or Non-Renewable Resource Exploitation: The Implication of Bounded Efficiency Improvements

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Assumptions and Calculus

Abstract

The paper presents the positive and normative analysis of endogenous R&D investment on two types of resources: renewable and non-renewable. The specificity of the paper has to be found in the following assumption: resource-specific R&D investment allows to increase the efficiency of resource exploitation, but the feasible efficiency improvements are globally bounded from above. We make this crucial assumption to take into account the second principle of thermodynamics. It has important implications for the analysis.

First, the system does not admit any balanced path for the decentralized economy, because the growth rate in the efficiency of exploitation of any resource cannot be constant. Both effort in R&D sectors is decreasing since the marginal reward to R&D activity declines as the upper bound approaches. Second, in the decentralized economy, R&D effort is decreasing in the resource-specific efficiency level, because the marginal reward to R&D activity declines as the upper bound approaches. As a consequence, the share of resources devoted to R&D falls asymptotically towards zero. Third, the finiteness of efficiency improvements together with that of the non-renewable resource supply, imply that the initial conditions determine the qualitative feature of the transition path. In this case, R&D firms make choices taking into account a limited time horizon since successive innovations make patents obsolete. R&D tends to concentrate in the resource sector where the demand is the largest

and where the scope for efficiency improvement is the highest. When the first effect dominates, the technological gap increases, giving rise to

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possible unbalanced rushes of R&D on one resource. When the initial efficiency gap is favorable to the non-renewable resource, our model predicts that R&D activity focuses on this resource first, causing the gap to increase. Eventually, as the growth of the resource rent reduces the competitiveness of the non-renewable resource, R&D focuses gradually and eventually exclusively on the renewable substitute.

This development is in contrast with the qualitative features of the optimal path of technological change. Because of the limited availability of the non renewable resource, whenever efficiency improvement on this resource is worth, it is preferable from a social point of view to obtain it as soon as possible. In fact, the improved efficiency can be applied to a larger resource stock if it is obtained early rather than late.

The timing and the relative size of R&D effort are typically different in the decentralized and the centralized economies.

Keywords: Direction of technological change, Energy augmenting technological progress, Resource substitution.
JEL codes: Q30, Q40.

1 Introduction

The paper presents the positive and normative analysis of endogenous R&D investment on two types of resources: renewable and non-renewable. Resources energy efficiencies are far different from each other: for instance fossil fuel energy (as oil or natural gas) for depletable resources; hydropower, wind energy, solar energy (as photovoltaic or thermal), biomass and geothermal energy for expendable energy. In a historical context, economies see their total energy use increases as they are developing but they also know a transition path between renewable to non-renewable energy resources. The latter path appears as (of course) the non-renewable resources deplete and as (also) the R&D activities improve the efficiency of back-stop technology (the renewable resource which efficiency is high). The bi-sectoral model of R&D we propose is composed of quality innovations devoted to each types of resources: the purpose of R&D firms is to improve either the energy efficiency of the non-renewable resource or that of the renewable resource; both energy forms can be used simultaneously (differentiated inputs assumption).

The specificity of the paper has to be found in the following assumption: resource-specific R&D investment allows to increase the efficiency of resource exploitation, but the feasible efficiency improvements are globally bounded from above. We make this crucial assumption to take into account the second principle of thermodynamics. Then it leads to differentiate the traditional quality innovations for capital intermediates goods (as depicted by Aghion-Howitt's [1992] vertical growth model) from the innovations for energy inputs. It has important implications for the analysis. The well-known phenomenon that efficiency of both natural resources is bounded below unity is absent from theoretical literature although it proves quite relevant from thermodynamics principles.

We study both optimal dynamics and market (decentralized economy) dynamics. The issues we tend to answer are the following:

- Which R&D sector will be the first to reach a zero labor share (R&D effort)? Non-renewable resource efficiency or Renewable resource efficiency improvement?
- Does the transition between non-renewable resource to renewable resource occur abruptly? or Do R&D activities prepare the renewable energy (the backstop technology) toward its future role in order to ensure a smooth transition?
- Is there a difference of "R&D exhaustion timing" between the optimum and the decentralized economies?

In section 2 we study the optimal dynamics. In section 3 we investigate the decentralized economy with two models: no continuum and continuum of intermediates goods models. We conclude in section 4. In section 5 (Appendices) we mainly develop the optimal and the decentralized economies without substitute resource.

2 Social Planner: Fossil and Renewable Substitute

2.1 The Model Framework

Now we study the normative analysis of endogenous R&D investment on two types of resources (renewable resource (RR) and non-renewable resource (NRR)) as two types of factors. The resource-specific R&D investment allows to increase the efficiency of resource exploitation, but the feasible efficiency improvements are globally bounded from above:

$$Y_t = L_t^{1-\alpha}\left[\left[(1-v_t)^{\frac{1-\alpha}{\alpha}}\,x_t\right]^{\beta}\left[(1-u_t)^{\frac{1-\alpha}{\alpha}}\,s_t\right]^{1-\beta}\right]^{\alpha} \qquad (1)$$

with $(1-v_t)$ the efficiency index for the NRR where v_t is the efficiency waste, $v_t \in \left]0,\, 1\right]$; $(1-u_t)$ the efficiency index for the RR with $u_t \in \left]0,\, 1\right]$;

and

$$\begin{cases} \dot{v}_t = -\lambda_1 n_{1t} v_t \\ \dot{u}_t = -\lambda_2 n_{2t} u_t \end{cases}$$

In that first version of our work, we study the trivial results of the *Cobb-Douglas* function (1) and no inter-sectoral R&D spillover. The *CES* function would prove more interesting.

When the share of the renewable goes to zero ($\beta \to 1$), the model reduces to the one in the Appendix 1, since $Y_t = L_t^{1-\alpha}\left(1-v_t\right)^{1-\alpha} {x_t}^{\alpha}$.

The problem has 4 controls (x, s, n_1, n_2), 3 states (X, v, and u) with 3 (positive) costate variables (μ, κ, and ζ respectively):

$$\left\{ \begin{array}{l} \max\limits_{\{(n_t, x_t)\}_{t=0}^{+\infty}} \int\limits_0^{+\infty} e^{-\rho t} u\left(C_t\right) dt \\ \text{s.t.:} \left| \begin{array}{l} C_t = Y_t - c s_t \\ \dot{X}_t = -x_t \\ \dot{v}_t = -\lambda_1 n_{1t} v_t \\ \dot{u}_t = -\lambda_2 n_{2t} u_t \end{array} \right. \end{array} \right.$$

Taking into account that labor is available in fixed supply N, the current value *Hamiltonian* is:

$$\begin{aligned} H_c &= u(Y_t - c s_t) + \mu_t \dot{X}_t - \kappa_t \dot{v}_t - \zeta_t \dot{u}_t \\ \text{''} &= u\left(\left[\left(N - n_{1t} - n_{2t}\right) \left(1 - v_t\right)^{\beta} \left(1 - u_t\right)^{1-\beta} \right]^{1-\alpha} \left[x_t^{\beta} s_t^{1-\beta} \right]^{\alpha} - c.s_t \right) \\ & \quad -\mu_t x_t + \kappa_t \lambda_1 n_{1t} v_t + \zeta_t \lambda_2 n_{2t} u_t \end{aligned}$$

The following are the first order conditions with respect to x, s, n_1 and n_2 respectively:

$$\left\{ \begin{array}{l} \mu_t = u'(C_t) \alpha \beta \frac{Y_t}{x_t} \\ c = \alpha \left(1 - \beta\right) \frac{Y_t}{s_t} \\ \left| \begin{array}{l} u'(C_t)\left(1 - \alpha\right) \frac{Y_t}{N - n_{1t} - n_{2t}} \leq \kappa_t \lambda_1 v_t \\ \quad \text{with } n_{1t} \left(u'(C_t)\left(1 - \alpha\right) \frac{Y_t}{N - n_{1t} - n_{2t}} - \kappa_t \lambda_1 v_t \right) = 0 \end{array} \right. \\ \left| \begin{array}{l} u'(C_t)\left(1 - \alpha\right) \frac{Y_t}{N - n_{1t} - n_{2t}} \leq \zeta_t \lambda_2 u_t \\ \quad \text{with } n_{2t} \left(u'(C_t)\left(1 - \alpha\right) \frac{Y_t}{N - n_{1t} - n_{2t}} - \zeta_t \lambda_2 u_t \right) = 0 \end{array} \right. \end{array} \right. \tag{2}$$

The first two conditions on resource allocation together give:

$$x_t = \frac{\beta}{1 - \beta} c \frac{s_t}{\mu_t} u'(C_t) \tag{3}$$

2.2 Interior Solution

In the case of an interior solution ($n_{1t} > 0$ and $n_{2t} > 0$) the last two conditions on R&D effort equate the marginal product of labor in the final sector to its marginal return in the two R&D sectors. Therefore equating the latter, we obtain the following arbitrage across R&D sectors:

$$\lambda_1 \kappa_t v_t = \lambda_2 \zeta_t u_t \tag{4}$$

The Euler condition with respect to the 3 states X, $(-v)$, and $(-u)$, respectively, are:

$$\begin{aligned}
\dot{\mu}_t &= \rho\mu_t \qquad \Leftrightarrow \qquad \mu_t = \mu_0 e^{\rho t} \\
\frac{\dot{\kappa}_t}{\kappa_t} &= \rho + \lambda_1 n_{1t} - \beta\left(1-\alpha\right)\frac{Y_t}{\kappa_t\left(1-v_t\right)}u'(C_t) \\
\frac{\dot{\zeta}_t}{\zeta_t} &= \rho + \lambda_2 n_{2t} - \left(1-\beta\right)\left(1-\alpha\right)\frac{Y_t}{\zeta_t\left(1-u_t\right)}u'(C_t)
\end{aligned}$$

i.e., using (32) and (33) in the case of an interior solution:

$$\dot{\mu}_t = \rho\mu_t \qquad \Leftrightarrow \qquad \mu_t = \mu_0 e^{\rho t} \tag{5}$$

$$\frac{\dot{\kappa}_t}{\kappa_t} = \rho + \lambda_1 n_{1t}\left(1 - \beta\frac{N - n_{1t} - n_{2t}}{n_{1t}}\frac{v_t}{1-v_t}\right) \tag{6}$$

$$\frac{\dot{\zeta}_t}{\zeta_t} = \rho + \lambda_2 n_{2t}\left(1 - (1-\beta)\frac{N - n_{1t} - n_{2t}}{n_{2t}}\frac{u_t}{1-u_t}\right) \tag{7}$$

Logdifferentiating (35) and substituting using (37) and (38):

$$\begin{aligned}
0 &= \frac{\dot{\kappa}_t}{\kappa_t} - \frac{\dot{\zeta}_t}{\zeta_t} + \frac{\dot{v}_t}{v_t} - \frac{\dot{u}_t}{u_t} \\
0 &= -\lambda_1 n_{1t}\beta\frac{N - n_{1t} - n_{2t}}{n_{1t}}\frac{v_t}{1-v_t} + \lambda_2 n_{2t}(1-\beta)\frac{N - n_{1t} - n_{2t}}{n_{2t}}\frac{u_t}{1-u_t} \\
0 &= (N - n_{1t} - n_{2t})\left[-\beta\lambda_1\frac{v_t}{1-v_t} + (1-\beta)\lambda_2\frac{u_t}{1-u_t}\right]
\end{aligned}$$

i.e.

$$\beta\lambda_1\frac{v_t}{1-v_t} = (1-\beta)\lambda_2\frac{u_t}{1-u_t} \tag{8}$$

i.e.

$$u_t = \frac{\beta\lambda_1 v_t}{(1-\beta)\lambda_2(1-v_t) + \beta\lambda_1 v_t} \tag{9}$$

Recall that these last two conditions hold for the interior solution, they do not have to hold at the initial date. Equation (9) defines the optimal level of renewable waste index for as function of the waste index for the non renewable. We have:

$$\frac{\partial u_t}{\partial v_t} = \frac{\beta(1-\beta)\lambda_1\lambda_2}{\left[(1-\beta)\lambda_2(1-v_t) + \beta\lambda_1 v_t\right]^2}$$

$$\begin{aligned}
\frac{\partial^2 u_t}{\partial v_t^2} &= \left(\frac{1}{\frac{(1-\beta)\lambda_2}{[(1-\beta)\lambda_2 - \beta\lambda_1]} - v_t}\right)\frac{2\beta(1-\beta)\lambda_1\lambda_2}{\left[(1-\beta)\lambda_2 - \left[(1-\beta)\lambda_2 - \beta\lambda_1\right]v_t\right]^2} \\
\frac{\partial^2 u_t}{\partial v_t^2} &\leq 0 \quad \Leftrightarrow \quad \beta \geq \frac{\lambda_2}{\lambda_1 + \lambda_2}
\end{aligned}$$

Logdifferentiating (8) and using (9):

$$\frac{\dot{v}_t}{v_t(1-v_t)} = \frac{\dot{u}_t}{u_t(1-u_t)} \Leftrightarrow \frac{\dot{v}_t}{v_t} = \frac{\dot{u}_t}{u_t}\left((1-v_t) + \frac{\beta}{1-\beta}\frac{\lambda_1}{\lambda_2}v_t\right)$$

from which we deduce:

$$\frac{n_{1t}}{n_{2t}} = \frac{\lambda_2}{\lambda_1}(1-v_t) + \frac{\beta}{1-\beta}v_t \tag{10}$$

From (31) we have that Y/s is constant, i.e. $\dot{Y}/Y = \dot{s}/s$. Logdifferentiating the production function as written in H (and dropping t), we obtain:

$$\frac{\dot{Y}}{Y} = (1-\alpha)\left[-\frac{\dot{n}_1+\dot{n}_2}{N-n_1-n_2} - \beta\frac{\dot{v}}{1-v} - (1-\beta)\frac{\dot{u}}{1-u}\right] + \alpha\left[\beta\frac{\dot{x}}{x} + (1-\beta)\frac{\dot{s}}{s}\right]$$

Logdifferentiating (34) and using (36):

$$\frac{\dot{x}}{x} = \frac{\dot{s}}{s} - \frac{\dot{\mu}}{\mu} - \frac{1}{\sigma}\frac{\dot{C}}{C} = \frac{\dot{s}}{s} - \rho - \frac{1}{\sigma}\frac{\dot{C}}{C}$$

substituting in the previous expression:

$$\frac{\dot{Y}}{Y} = (1-\alpha)\left[-\frac{\dot{n}_1+\dot{n}_2}{N-n_1-n_2} - \beta\frac{\dot{v}}{1-v} - (1-\beta)\frac{\dot{u}}{1-u}\right] + \alpha\beta\left(-\rho - \frac{1}{\sigma}\frac{\dot{C}}{C}\right) + \alpha\frac{\dot{s}}{s}$$

and taking into account that $\dot{Y}/Y = \dot{C}/C = \dot{s}/s$:

$$\begin{aligned}\left(1+\frac{\alpha\beta}{1-\alpha}\frac{1}{\sigma}\right)\frac{\dot{Y}}{Y} &= \frac{-(\dot{n}_1+\dot{n}_2)}{N-n_1-n_2} - \beta\frac{\dot{v}}{1-v} - (1-\beta)\frac{\dot{u}}{1-u} - \frac{\alpha\beta\rho}{1-\alpha} \\ " &= \frac{-(\dot{n}_1+\dot{n}_2)}{N-n_1-n_2} + \frac{v\beta\lambda_1 n_1}{1-v} + \frac{u(1-\beta)\lambda_2 n_2}{1-u} - \frac{\alpha\beta\rho}{1-\alpha}\end{aligned} \tag{11}$$

Next, logdifferentiating (32) in the case of an interior solution:

$$\left(1-\frac{1}{\sigma}\right)\frac{\dot{Y}}{Y} + \frac{\dot{n}_1+\dot{n}_2}{N-n_1-n_2} = \frac{\dot{\kappa}}{\kappa} + \frac{\dot{v}}{v}$$

substituting for (37):

$$\left(1-\frac{1}{\sigma}\right)\frac{\dot{Y}}{Y} + \frac{\dot{n}_1+\dot{n}_2}{N-n_1-n_2} = \rho - \beta\lambda_1(N-n_1-n_2)\frac{v}{1-v}$$

combining this result with (**??**) we have:

$$\begin{aligned}\left(1+\frac{\alpha\beta}{1-\alpha}\right)\frac{1}{\sigma}\frac{\dot{Y}}{Y} &= \left(\beta\lambda_1 n_1\frac{v}{1-v} + (1-\beta)\lambda_2 n_2\frac{u}{1-u}\right) - \frac{\alpha\beta}{1-\alpha}\rho \\ &\quad -\rho + \beta\lambda_1(N-n_1-n_2)\frac{v}{1-v} \\ " &= \beta\lambda_1(N-n_2)\frac{v}{1-v} + (1-\beta)\lambda_2 n_2\frac{u}{1-u} - \left(1+\frac{\alpha\beta}{1-\alpha}\right)\rho\end{aligned}$$

i.e.

$$\frac{\dot{Y}}{Y} = \sigma\left[\frac{1-\alpha}{\alpha\beta+1-\alpha}\left(\beta\lambda_1(N-n_2)\frac{v}{1-v} + (1-\beta)\,\lambda_2 n_2\frac{u}{1-u}\right) - \rho\right]$$

and using (9):

$$\frac{\dot{Y}}{Y} = \sigma\left[\frac{1-\alpha}{\alpha\beta+1-\alpha}\beta\lambda_1\frac{v}{1-v}N - \rho\right] \tag{12}$$

Then we have

$$\begin{aligned}
\frac{\dot{n}_1+\dot{n}_2}{N-n_1-n_2} &= \rho - \beta\lambda_1(N-n_1-n_2)\frac{v}{1-v} - \left(1-\frac{1}{\sigma}\right)\frac{\dot{Y}}{Y} \\
\text{"} &= \sigma\rho - \beta\lambda_1\frac{v}{1-v}\left[MN-n_1-n_2\right]
\end{aligned}$$

Define $n = n_1 + n_2$ and $\phi = n_1/n$, (10) then gives:

$$\frac{\phi}{1-\phi} = \frac{\lambda_2}{\lambda_1}(1-v) + \frac{\beta}{1-\beta}v$$

Interior solution system:

$$\begin{aligned}
\frac{\dot{v}}{v} &= -\lambda_1\phi n \\
\frac{\dot{u}}{u} &= -\lambda_2(1-\phi)n \\
\phi &= \frac{\frac{\lambda_2}{\lambda_1}(1-v) + \frac{\beta}{1-\beta}v}{1+\frac{\lambda_2}{\lambda_1}(1-v) + \frac{\beta}{1-\beta}v} \\
\frac{\dot{n}}{n} &= \frac{N-n}{n}\left[\sigma\rho - \beta\lambda_1\frac{v}{1-v}\left[MN-n\right]\right]
\end{aligned}$$

$$\dot{n} = 0 \Leftrightarrow \begin{cases} n = N \\ \text{or} \\ n = MN - \frac{\sigma\rho}{\beta\lambda_1}\frac{1-v_t}{v_t} \end{cases}$$

$$\dot{v} = \dot{u} = 0 \Leftrightarrow n = 0$$

Stationary solution:

$$\begin{cases} n^* = 0 \\ v^* = \frac{1}{1+\frac{\beta\lambda_1}{\sigma\rho}MN} \\ u^* = \frac{1}{1+\frac{(1-\beta)\lambda_2}{\sigma\rho}MN} \\ \phi^* = \frac{\frac{\lambda_2}{\lambda_1}(1-v^*)+\frac{\beta}{1-\beta}v^*}{1+\frac{\lambda_2}{\lambda_1}(1-v^*)+\frac{\beta}{1-\beta}v^*} \\ \left(\frac{\dot{Y}}{Y}\right)^* = -\sigma\rho\frac{\alpha\beta}{\alpha\beta+\sigma(1-\alpha)} \end{cases}$$

2.3 Corner Solutions

Solution with $n_{1t} > 0$ and $n_{2t} = 0$, $\dot{u}_t = 0$, $u_t = u_0$ with

$$u_0 < \frac{\beta\lambda_1 v_0}{(1-\beta)\lambda_2(1-v_0)+\beta\lambda_1 v_0}$$

Then we have:

$$\begin{aligned} u'(C_t)\,(1-\alpha)\,\frac{Y_t}{N-n_{1t}} &= \lambda_1\kappa_t v_t \\ \lambda_1\kappa_t v_t &< \lambda_2\zeta_t u_t \end{aligned}$$

i.e. during this initial phase investing in u is characterized by:

$$\underbrace{-u'(C_t)\,(1-\alpha)\,\frac{Y_t}{N-n_{1t}-n_{2t}}}_{\text{marginal cost}} > \underbrace{-\lambda_2\zeta_t u_0}_{\text{marg. reward}}$$

Most of the analysis carries over to this case, bar the foc (33), (38) to (10) that suppose $n_2 > 0$. Yet output can be written as:

$$Y_t = \left[(N-n_{1t})\,(1-v_t)^{\beta}\,(1-u_0)^{1-\beta}\right]^{1-\alpha}\left[x_t^{\beta}s_t^{1-\beta}\right]^{\alpha}$$

while the growth rate is still given by equation (12).[1]

Dynamic system:

$$\begin{aligned} \dot{v} &= -\lambda_1 n_1 v \\ \dot{n}_1 &= (N-n_1)\left[\sigma\rho-\beta\lambda_1\frac{v}{1-v}\left[MN-n_1\right]\right] \end{aligned}$$

Stationary solution:

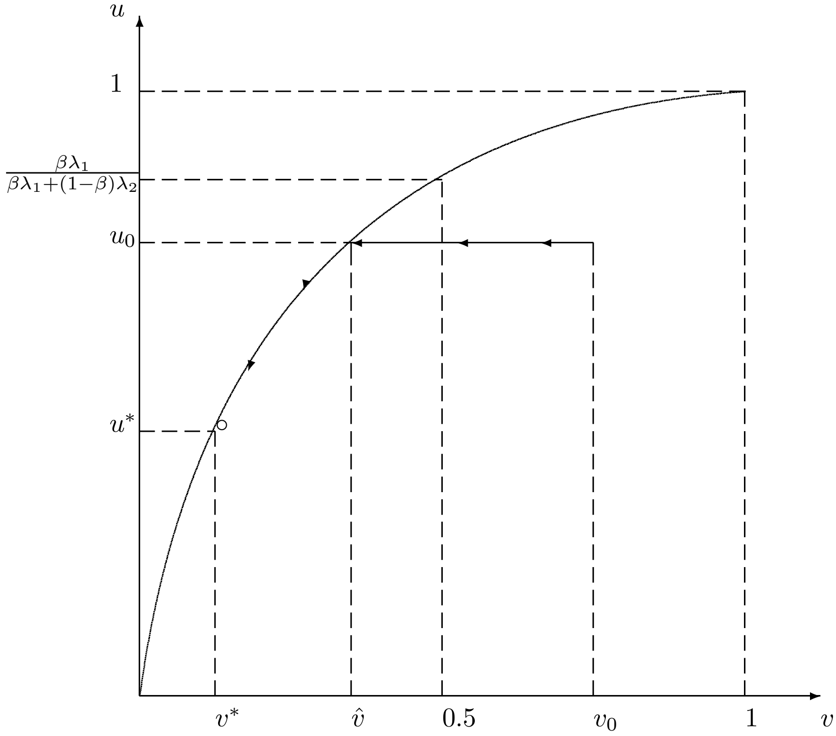
$$\begin{cases} n_1^* = 0 \\ v^* = \frac{1}{1+\frac{\beta\lambda_1}{\sigma\rho}MN} \end{cases}$$

with $M = 1-(1-\sigma)\frac{1-\alpha}{1-\alpha+\alpha\beta} = \frac{\alpha\beta+\sigma(1-\alpha)}{\alpha\beta+1-\alpha}$

2.4 Full dynamics

As the figures (1) and (2) show us, the optimal path is characterized by three phases (T_0, T_b and T^*):

[1] Actually (12) is obtained using (9) which holds for the interior solution. However, the epression right before (12) is obtained without assuming $n_2 > 0$. Hence setting $n_2 = 0$ we get again (12).



1. Unbalanced Path: At first investments in the resources which part in output function is the bigger. That is to say the elasticity in the *C-D* function. Historically, it is the case of the *NRR* and not the *RR* back stop technology;

2. Balanced Path: Simultaneous investments when expected invest profits are equalized in both R&D sectors;

3. Steady State: At some date, it stops because of marginal benefits falls compared to the costs of R&D. So the economy reaches a steady state.

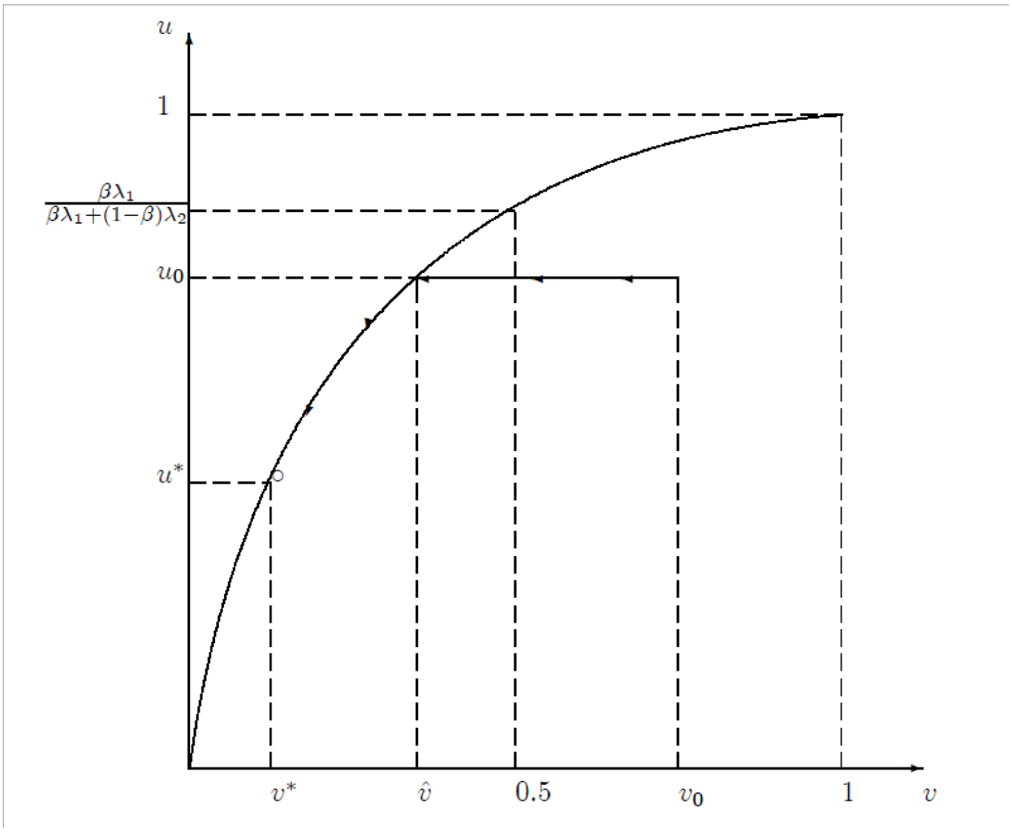

u
1
$\frac{\beta\lambda_1}{\beta\lambda_1+(1-\beta)\lambda_2}$
u_0
u^*
v^*
$\hat{v}$
0.5
v_0
1
v

Figure 1:

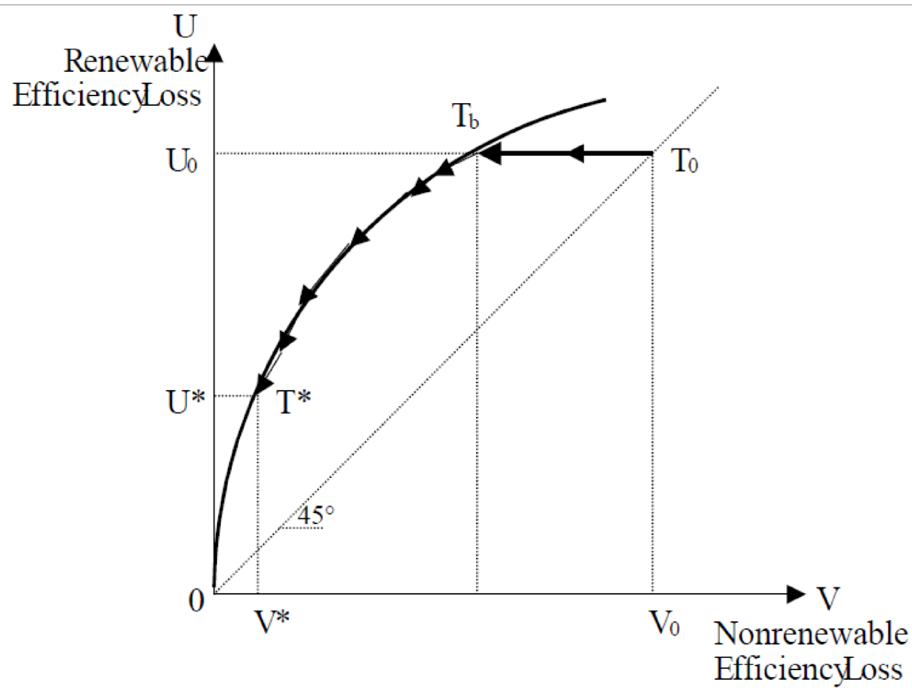


figure (2). Three Phases Optimal Path

Figure 2:

3 Decentralized Economy: Fossil and Renewable Substitute[2]

3.1 The Model Framework

The aggregate production function considered in the social planner's problem was:

$$\begin{aligned}
Y &= L^{1-\alpha}\left\{\left[(1-v)^{\frac{1-\alpha}{\alpha}}x\right]^{\beta}\left[(1-u)^{\frac{1-\alpha}{\alpha}}s\right]^{1-\beta}\right\}^{\alpha} \\
" &= L^{1-\alpha}(1-v)^{(1-\alpha)\beta}x^{\alpha\beta}(1-u)^{(1-\alpha)(1-\beta)}s^{\alpha(1-\beta)} \\
" &= L^{1-\alpha}\left[(1-v)\left(\frac{x}{1-v}\right)^{\alpha}\right]^{\beta}\left[(1-u)\left(\frac{s}{1-u}\right)^{\alpha}\right]^{1-\beta} \\
" &= L^{1-\alpha}\left[(1-v)(e_x)^{\alpha}\right]^{\beta}\left[(1-u)(e_s)^{\alpha}\right]^{1-\beta}
\end{aligned}$$

with $e_x = x/(1-v)$ and $e_s = x/(1-u)$.

For the decentralized economy, if we want to reproduce the problem solved for the social planner, we need to have continuous improvements in energy efficiency, $(1-v)$ and $(1-u)$. To ensure continuous changes in technological variables we should set up a model economy with a continuum of sectors, represented for instance by the following aggregate production function:

$$\begin{aligned}
Y &= L^{1-\alpha}\left[\int_0^1(1-v_j)(e_{xj})^{\alpha}dj\right]^{\beta}\left[\int_0^1(1-u_i)(e_{si})^{\alpha}di\right]^{1-\beta} \\
" &= L^{1-\alpha}\left[\int_0^1(1-v_j)^{1-\alpha}(x_j)^{\alpha}dj\right]^{\beta}\left[\int_0^1(1-u_i)^{1-\alpha}(s_i)^{\alpha}di\right]^{1-\beta}
\end{aligned}$$

which rise two questions:

- how can we justify the choice of equal masses (both unitary) of sub-sectors in x and in s?
- in general the aggregate production function does not correspond exactly to that of the social planner's problem, indeed:

$$\left[\int_0^1(1-v_j)^{1-\alpha}(x_j)^{\alpha}dj\right]^{\beta}\left[\int_0^1(1-u_i)^{1-\alpha}(s_i)^{\alpha}di\right]^{1-\beta} \neq \left[(1-v)^{1-\alpha}x^{\alpha}\right]^{\beta}\left[(1-u)^{1-\alpha}s^{\alpha}\right]^{1-\beta}$$

  because even if $x_j = x\ \forall j \in [0,1]$, still

$$\left(\int_0^1(1-v_j)^{1-\alpha}dj\right)^{\beta} > \int_0^1(1-v_j)^{1-\alpha}dj > \int_0^1(1-v_j)\,dj = 1-\int_0^1 v_j dj = 1-v$$

[2] Notes on decentralized economy (29/3/02 (note so sure)).
See Appendix 3 for the decentralized economy without substitute.

where: the first inequality holds because $1-v_j \in (0,1)\ \forall j$ and $1-\alpha<1$ $\Rightarrow \int_0^1 (1-v_j)^{1-\alpha} dj < 1$, together with $f(x) > x$ for $f(.)$ concave and $x \in (0,1)$; for the same reason $(1-v_j)^{1-\alpha} > (1-v_j)\ \forall j$ which explains the second inequality.

- fortunately, when $e_{xj} = e_x\ \forall j \in [0,1]$ and $e_{si} = e_s\ \forall i \in [0,1]$ the two aggregate configurations coincide:

$$\begin{aligned}
\left[\int_0^1 (1-v_j)(e_{xj})^\alpha dj\right]^\beta \left[\int_0^1 (1-u_i)(e_{si})^\alpha di\right]^{1-\beta} &= \left[e_x^\alpha \int_0^1 (1-v_j)\, dj\right]^\beta \left[e_s^\alpha \int_0^1 (1-u_i)\, di\right]^{1-\beta} \\
\text{"} &= \left[e_x^\alpha (1-v)\right]^\beta \left[e_s^\alpha (1-u)\right]^{1-\beta}
\end{aligned}$$

where we define $\int_0^1 (1-v_j)\, dj = 1-\int_0^1 v_j dj = 1-v$ and $\int_0^1 (1-u_i)\, di = 1-\int_0^1 u_i di = 1-u$

3.2 The Fake Problem: i.e. No Continuum

We can study the false problem of a decentralized economy with only two firms on the resource market, one for each resource. This ensures that the instantaneous representation of the aggregate production function coincides with the one retained for the social planner's problem. However, this is not accurate because with only two firms innovations are discontinuous.

The **final sector** representative (competitive) firm chooses L, e_x and e_s to maximize instantaneous profits

$$\Pi = L^{1-\alpha} \left[(1-v)(e_x)^\alpha\right]^\beta \left[(1-u)(e_s)^\alpha\right]^{1-\beta} - wL - p_x e_x - p_s e_s$$

the inverse demand functions are:

$$\begin{aligned}
w &= (1-\alpha)\frac{Y}{L} \\
p_x &= \alpha\beta\frac{Y}{e_x} \\
p_s &= \alpha(1-\beta)\frac{Y}{e_s}
\end{aligned}$$

The representative **monopoly on the x market**: demand of inputs $x^d = (1-v)\, e_x$, so that the marginal cost is $(1-v)\, q$, where q is the spot price of a unit of resource x. Instantaneous profits are therefore $\pi_x = [p_x - (1-v)\, q]\, e_x$. They are maximized by the monopolist taking into account the demand function:

$$\max_{e_x} \alpha\beta Y - (1-v)\, q e_x$$

foc: $(\alpha\beta)^2 Y/e_x = (1-v)\, q$

$$e_x = \left[(\alpha\beta)^2 L^{1-\alpha} \left(\frac{1-u}{1-v} \right)^{1-\beta} \frac{e_s^{\alpha(1-\beta)}}{q} \right]^{\frac{1}{1-\alpha\beta}}$$

The representative **monopoly on the s market**: demand of inputs $s^d = (1-u)\, e_s$, so that the marginal cost is $(1-u)\, c$, where c is the cost of a unit of resource s (asymmetry because there is no owner of resource s). Instantaneous profits are therefore $\pi_s = [p_s - (1-u)\, c]\, e_s$. They are maximized by the monopolist taking into account the demand function:

$$\max_{e_s} \alpha\,(1-\beta)\, Y - (1-u)\, c e_s$$

foc: $[\alpha\,(1-\beta)]^2\, Y/e_s = (1-u)\, c$

$$e_s = \left[[\alpha\,(1-\beta)]^2 L^{1-\alpha} \left(\frac{1-v}{1-u} \right)^{\beta} \frac{e_x^{\alpha\beta}}{c} \right]^{\frac{1}{1-\alpha(1-\beta)}}$$

Substitute for e_x using the previous result:

$$\begin{aligned} e_s &= \Gamma L \left(\frac{1-v}{1-u} \right)^{\beta} \left[c^{1-\alpha\beta} q^{\alpha\beta} \right]^{\frac{-1}{1-\alpha}} \\ e_x &= \Lambda L \left(\frac{1-u}{1-v} \right)^{1-\beta} \left[c^{\alpha(1-\beta)} q^{1-\alpha(1-\beta)} \right]^{\frac{-1}{1-\alpha}} \end{aligned}$$

where $\Gamma = \left\{ [\alpha\,(1-\beta)]^{1-\alpha\beta} (\alpha\beta)^{\alpha\beta} \right\}^{\frac{2}{1-\alpha}}$ and $\Lambda = \left(\Gamma^{\alpha(1-\beta)} (\alpha\beta)^2 \right)^{\frac{1}{1-\alpha\beta}} = \left\{ [\alpha\,(1-\beta)]^{\alpha(1-\beta)} (\alpha\beta)^{1-\alpha(1-\beta)} \right\}^{\frac{2}{1-\alpha}}$.

Taking into account these partial equilibrium values of intermediate goods' sales, we can compute aggregate output as:

$$Y = \Omega L\,(1-u)^{1-\beta}\,(1-v)^{\beta} \left(c^{1-\beta} q^{\beta} \right)^{\frac{-\alpha}{(1-\alpha)}}$$

where $\Omega = \Lambda^{\alpha\beta} \Gamma^{\alpha(1-\beta)} = \left\{ [\alpha\,(1-\beta)]^{\alpha(1-\beta)} (\alpha\beta)^{\alpha\beta} \right\}^{\frac{2}{1-\alpha}}$.

The profit for x producers are

$$\pi_q = qx = q\,(1-v)\, e_x = \Lambda L\,(1-u)^{1-\beta}\,(1-v)^{\beta} \left(c^{1-\beta} q^{\beta} \right)^{\frac{-\alpha}{(1-\alpha)}}$$

while:

$$cs = c\,(1-u)\, e_s = \Gamma L\,(1-u)^{1-\beta}\,(1-v)^{\beta} \left(c^{1-\beta} q^{\beta} \right)^{\frac{-\alpha}{(1-\alpha)}}$$

We can now compute the value of profits as function of L, q and technology:

$$\begin{aligned}\pi_x &= Y\left[\alpha\beta-(1-v)\frac{qe_x}{Y}\right] \\ " &= Y\left[\alpha\beta-\frac{\pi_q}{Y}\right] \\ " &= Y\left[\alpha\beta-\frac{\Lambda}{\Omega}\right] \\ " &= Y\left[\alpha\beta-\frac{\Lambda^{1-\alpha\beta}}{\Gamma^{\alpha(1-\beta)}}\right] \\ " &= \alpha\beta\left(1-\alpha\beta\right)Y\end{aligned}$$

and

$$\begin{aligned}\pi_s &= Y\left[\alpha\left(1-\beta\right)-\frac{cs_s}{Y}\right] \\ " &= Y\left[\alpha\left(1-\beta\right)-\frac{\Gamma}{\Omega}\right] \\ " &= Y\left[\alpha\left(1-\beta\right)-\frac{\Gamma^{1-\alpha(1-\beta)}}{\Lambda^{\alpha\beta}}\right] \\ " &= \alpha\left(1-\beta\right)\left[1-\alpha\left(1-\beta\right)\right]Y\end{aligned}$$

The value of an innovation in the x sector arrived at date t (with technology $\bar{v}_t$) is:

$$\begin{aligned}V_t^x &= \pi_{xt}\int_t^\infty e^{-\int_0^\tau(r_s+\lambda_1 n_{1s})ds}E\left(\frac{\pi_{x\tau}}{\pi_{xt}}\right)d\tau \\ " &= \alpha\beta\left(1-\alpha\beta\right)Y_t^{v+}\int_t^\infty e^{-\int_0^\tau(r_s+\lambda_1 n_{1s})ds}E\left(\frac{Y_\tau^{v+}}{Y_t^{v+}}\right)d\tau \\ " &= \alpha\beta\left(1-\alpha\beta\right)\Omega L_t\left(1-u_t\right)^{1-\beta}\left(1-\bar{v}_t\right)^\beta\left(c^{1-\beta}q_t^\beta\right)^{\frac{-\alpha}{(1-\alpha)}}\int_t^\infty e^{-\int_0^\tau(r_s+\lambda_1 n_{1s})ds}E\left(\frac{Y_\tau^{v+}}{Y_t^{v+}}\right)d\tau\end{aligned}$$

for the term $E\left(\frac{\pi_{x\tau}}{\pi_{xt}}\right)$ we notice that it depends on the expected rate of growth of output conditional on no innovations arriving on sector x. Innovations improve the overall energy efficiency, which tends to rise aggregate production, increasing demand for the x intermediate inputs (effect through $Y_\tau/Y_t = e^{\int_t^\tau g_{Ys}ds}$). The expected growth in demand for x intermediate inputs, runs also through the evolution of R&D activity, because this sector competes with the final sector for labor inputs. Hence, when R&D employment is expected to fall, employment in the final sector is expected to rise, boosting the productivity of x intermediate inputs (this effect is taken into account by the term Y_τ/Y_t too).

Similarly for the value of an innovation in sector s:

$$\begin{aligned}V_t^s &= \pi_{st}\int_t^\infty e^{-\int_0^\tau(r_s+\lambda_2 n_{2s})ds}E\left(\frac{\pi_{s\tau}}{\pi_{st}}\right)d\tau \\ " &= \alpha\left(1-\beta\right)\left[1-\alpha\left(1-\beta\right)\right]Y_t^{u+}\int_t^\infty e^{-\int_0^\tau(r_s+\lambda_2 n_{2s})ds}E\left(\frac{Y_\tau^{u+}}{Y_t^{u+}}\right)d\tau\end{aligned}$$

The equilibrium on the labor market requires: $\forall t \; L_t + n_{1t} + n_{2t} = N$ and

$$w_t = (1-\alpha)\frac{Y_t}{(N - n_{1t} - n_{2t})} = \lambda_1 V_t^x = \lambda_2 V_t^s$$

i.e.

$$\begin{aligned}
\frac{(1-\alpha)}{(N-n_{1t}-n_{2t})} &= \lambda_1 \alpha\beta\left(1-\alpha\beta\right)\frac{Y_t^{v+}}{Y_t}\int_t^{\infty} e^{-\int_t^{\tau}(r_s+\lambda_1 n_{1s})ds} E\left(\frac{Y_\tau^{v+}}{Y_t^{v+}}\right)d\tau \\
" &= \lambda_2 \alpha\left(1-\beta\right)\left[1-\alpha\left(1-\beta\right)\right]\frac{Y_t^{u+}}{Y_t}\int_t^{\infty} e^{-\int_t^{\tau}(r_s+\lambda_2 n_{2s})ds} E\left(\frac{Y_\tau^{u+}}{Y_t^{u+}}\right)d\tau
\end{aligned}$$

3.3 A Continuum of Intermediate Goods

The aggregate production function is:

$$Y = L^{1-\alpha}\left\{\int_0^1 (1-v_j)\left(e_{xj}\right)^{\alpha} dj\right\}^{\beta}\left\{\int_0^1 (1-u_i)\left(e_{si}\right)^{\alpha} di\right\}^{1-\beta}$$

Instantaneous profits of the fictitious **final sector** firm are:

$$\Pi = L^{1-\alpha}\left\{\int_0^1 (1-v_j)\left(e_{xj}\right)^{\alpha} dj\right\}^{\beta}\left\{\int_0^1 (1-u_i)\left(e_{si}\right)^{\alpha} di\right\}^{1-\beta} - wL - \int_0^1 p_{xj}e_{xj}dj - \int_0^1 p_{si}e_{si}di$$

hence the inverse demand functions are, $\forall i, j$:

$$\begin{aligned}
w &= (1-\alpha)\frac{Y}{L} \\
p_{xj} &= \alpha\beta\frac{Y}{X}\left(1-v_j\right)e_{xj}^{\alpha-1} \\
p_{si} &= \alpha\left(1-\beta\right)\frac{Y}{S}\left(1-u_i\right)e_{si}^{\alpha-1}
\end{aligned}$$

where we define $X = \int_0^1 (1-v_j)(e_{xj})^{\alpha} dj$ and $S = \int_0^1 (1-u_i)(e_{si})^{\alpha} di$.

The representative **monopolist** in the x sector maximizes instantaneous profits taking as given Y/X (because it is an aggregate variable over which a single firm has no influence). The unit cost is proportional to the efficiency of its product, since the production technology $e_{xj} = x/(1-v_j)$ implies unit cost $(1-v_j)\,q$, where q is the spot price of the non renewable resource. The monopolist takes into account the from of the demand function it is confronted to:

$$\begin{aligned}
\max_{e_{xj}} \pi_{xj} &= \left[p_{xj} - (1-v_j)\,q\right]e_{xj} \\
" &= \alpha\beta\frac{Y}{X}\left(1-v_j\right)e_{xj}^{\alpha} - \left(1-v_j\right)qe_{xj}
\end{aligned}$$

which gives partial equilibrium sales, the price rule and profits:

$$\begin{aligned}
e_{xj} &= \left(\frac{\alpha^2\beta}{q_t}\frac{Y}{X}\right)^{\frac{1}{1-\alpha}} \equiv e_x(t) \qquad \forall j \in [0,1] \\
p_{xj} &= \frac{(1-v_j)\,q}{\alpha} \\
\pi_{xj} &= \frac{1-\alpha}{\alpha}(1-v_j)\,qe_x \\
" &= \frac{1-\alpha}{\alpha}\left(\alpha^2\beta\right)^{\frac{1}{1-\alpha}}\left(\frac{Y}{X}\right)^{\frac{1}{1-\alpha}}(1-v_j)\,q_t^{\frac{-\alpha}{1-\alpha}}
\end{aligned}$$

Notice that the result implies that the aggregate representation of the decentralized economy coincides with the one retained for the analysis of the social planner's problem, given that intermediate goods are sold uniformly across sectors. It follows that $X = \int_0^1 (1-v_j)(e_{xj})^{\alpha}\, dj = e_x^{\alpha}(1-v)$ and substituting[3]

$$\begin{aligned}
e_{xj} &= \left[\left(\frac{\alpha^2\beta}{q_t}\right)\frac{Y}{e_x^{\alpha}(1-v)}\right]^{\frac{1}{1-\alpha}} \\
e_x &= \frac{\alpha^2\beta}{(1-v)\,q_t}Y_t
\end{aligned}$$

Therefore:

$$\begin{aligned}
\pi_{xj} &= \frac{1-\alpha}{\alpha}(1-v_j)\,qe_x \\
" &= \alpha\beta(1-\alpha)\left(\frac{1-v_j}{1-v}\right)Y_t
\end{aligned}$$

Similarly in the case of the monopolist in the s sector we get:

$$\begin{aligned}
e_{si} &= \left[\left(\frac{\alpha^2(1-\beta)}{c}\right)\frac{Y}{S}\right]^{\frac{1}{1-\alpha}} \equiv e_s \quad \forall i \in [0,1] \\
e_s &= \frac{\alpha^2(1-\beta)}{(1-u)\,c}Y \\
\pi_{sj} &= \alpha(1-\beta)(1-\alpha)\left(\frac{1-u_j}{1-u}\right)Y_t
\end{aligned}$$

Therefore we can compute aggregate output as follows:

$$\begin{aligned}
Y &= L^{1-\alpha}X^{\beta}S^{1-\beta} = L^{1-\alpha}\left[e_x^{\alpha}(1-v)\right]^{\beta}\left[e_s^{\alpha}(1-u)\right]^{1-\beta} \\
" &= L^{1-\alpha}\left(\frac{\alpha^2\beta}{(1-v)\,q_t}Y_t\right)^{\alpha\beta}(1-v)^{\beta}\left(\frac{\alpha^2(1-\beta)}{(1-u)\,c}Y_t\right)^{\alpha(1-\beta)}(1-u)^{1-\beta} \\
" &= \left[\left(\alpha^2\beta\right)^{\beta}\left(\alpha^2(1-\beta)\right)^{1-\beta}\right]^{\frac{\alpha}{1-\alpha}} L(1-v)^{\beta}(1-u)^{1-\beta}\left(q_t^{\beta}c^{1-\beta}\right)^{\frac{-\alpha}{1-\alpha}}
\end{aligned}$$

[3] The profit of non renewable resource suppliers is $\pi_q = q\int x_j dj = q\int (1-v_j)\,e_{xj}dj = \alpha^2\beta Y_t$

We also have that:
$$\frac{x_t}{s_t} = \frac{e_{xt}\left(1-v_t\right)}{e_{st}\left(1-u_t\right)} = \frac{\beta}{1-\beta}\frac{c}{q_t}$$
Define the instantaneous probability of survival in sector $j = x, s$ as $\sigma_j n_{jt}$, to distinguish it from the growth rate of the technological parameters, $g_v = -\lambda_1 n_{1t}$.

The **value of an innovation** arrived at date t on sector x equals:

$$\begin{aligned}
V^x\left(t,\bar{v}_t\right) &= \pi_x\left(t,\bar{v}_t\right)\int_t^\infty e^{-\int_t^\tau [r(s)+\sigma_v n_v(s)]ds}\frac{\pi_x\left(\tau,\bar{v}_t\right)}{\pi_x\left(t,\bar{v}_t\right)}d\tau \\
\text{''} &= \pi_x\left(t,\bar{v}_t\right)\int_t^\infty e^{-\int_t^\tau [r(s)+\sigma_v n_v(s)]ds}\frac{\frac{1-\alpha}{\alpha}\left(1-\bar{v}_t\right)q_\tau e_{x\tau}}{\frac{1-\alpha}{\alpha}\left(1-\bar{v}_t\right)q_t e_{xt}}d\tau \\
\text{''} &= \pi_x\left(t,\bar{v}_t\right)\int_t^\infty e^{-\int_t^\tau [r(s)+\sigma_{v1} n_v(s)]ds}\frac{q_\tau\left(1-v_t\right)q_t Y_\tau}{q_t\left(1-v_\tau\right)q_\tau Y_t}d\tau \\
\text{''} &= \pi_x\left(t,\bar{v}_t\right)\int_t^\infty e^{-\int_t^\tau [r(s)+\sigma_v n_v(s)]ds}\left(\frac{Y_\tau}{1-v_\tau}\Big/\frac{Y_t}{1-v_t}\right)d\tau \\
\text{''} &= \pi_x\left(t,\bar{v}_t\right)\int_t^\infty \underbrace{e^{-\int_t^\tau [r(s)+\sigma_v n_v(s)]ds}}_{\text{discounting and survival}}\underbrace{e^{\int_t^\tau \frac{\lambda_v n_v(s)v(s)}{1-v(s)}ds}}_{obsolescence}\underbrace{e^{\int_t^\tau g_Y(s)ds}}_{\text{general demand}}d\tau
\end{aligned}$$

$(g_{(1-v)} = \frac{-\dot{v}}{1-v} = \frac{\lambda_v n_v v}{1-v})$. We can use this result to write the R&D arbitrage condition in the x sector:

$$\begin{aligned}
\left(1-\alpha\right)\frac{Y_t}{L_t} &= w_t = \sigma_v V^x\left(t,\bar{v}_t\right) \\
\left(1-\alpha\right)\frac{Y_t}{L_t} &= \sigma_v \pi_x\left(t,\bar{v}_t\right)\int_t^\infty e^{-\int_t^\tau [r(s)+\sigma_v n_v(s)]ds}\left(\frac{Y_\tau}{1-v_\tau}\Big/\frac{Y_t}{1-v_t}\right)d\tau \\
\frac{\left(1-\alpha\right)}{L_t} &= \sigma_v\frac{\pi_x\left(t,\bar{v}_t\right)}{Y_t}\int_t^\infty e^{-\int_t^\tau [r(s)+\sigma_v n_v(s)]ds}\left(\frac{Y_\tau}{1-v_\tau}\Big/\frac{Y_t}{1-v_t}\right)d\tau
\end{aligned}$$

and using previous results

$$\frac{\pi_x\left(t,\bar{v}_t\right)}{Y_t} = \frac{\frac{1-\alpha}{\alpha}\left(1-\bar{v}_t\right)q_t e_{xt}}{Y_t} = \frac{\frac{1-\alpha}{\alpha}\left(1-\bar{v}_t\right)q_t}{Y_t}\frac{\alpha^2\beta}{\left(1-v_t\right)q_t}Y_t = \left(1-\alpha\right)\alpha\beta\frac{1-\bar{v}_t}{1-v_t}$$

thus

$$\begin{aligned}
\frac{1}{L_t} &= \sigma_v \alpha\beta \frac{1-\bar{v}_t}{1-v_t} \int_t^{\infty} e^{-\int_t^{\tau}[r(s)+\sigma_v n_v(s)]ds} \left(\frac{Y_\tau}{1-v_\tau} \Big/ \frac{Y_t}{1-v_t} \right) d\tau \\
" &= \sigma_v \alpha\beta \int_t^{\infty} e^{-\int_t^{\tau}[r(s)+\sigma_v n_v(s)]ds} \left(\frac{1-\bar{v}_t}{1-v_\tau} \frac{Y_\tau}{Y_t} \right) d\tau \\
" &= \sigma_v \alpha\beta \int_t^{\infty} e^{-\int_t^{\tau}[r(s)+\sigma_v n_v(s)]ds} \left(\frac{1-\bar{v}_t}{1-v_\tau} \right) e^{\int_t^{\tau} g_Y(s)ds} d\tau
\end{aligned}$$

We want to derive a **dynamic system** for the variables $n = n_v + n_u$, $\phi = \frac{n_v}{n_v+n_u}$, v and u. We could use the Bellman equations:

$$\begin{cases} \frac{\dot{V}_x}{V_x} + \frac{\pi_x}{V_x} = r + \sigma_v n_v \\ \frac{\dot{V}_s}{V_s} + \frac{\pi_s}{V_s} = r + \sigma_u n_u \end{cases}$$

From the arbitrage condition we have that the value of an innovating firm at date t that adopts technology $\bar{v}_t$, satisfies, if $n_v > 0$:

$$\sigma_v V_x(t, \bar{v}_t) = w_t = (1-\alpha)\frac{Y_t}{L_t}$$

using the value of π derived above:

$$\frac{\pi_x(t,\bar{v}_t)}{V_x(t,\bar{v}_t)} = \sigma_v \alpha\beta L_t \frac{1-\bar{v}_t}{1-v_t}$$

Next, applying the same procedure as above to obtain the value at date t of a firm with technology v_j we have:

$$V^x(t, v_j) = \pi_x(t, v_j) \int_t^{\infty} e^{-\int_t^{\tau}[r(s)+\sigma_1 n_1(s)]ds} e^{\int_t^{\tau} \frac{\lambda_1 n_1(s) v(s)}{1-v(s)} ds} e^{\int_t^{\tau} g_Y(s)ds} d\tau$$

hence $\forall j$:

$$\frac{\pi_x(t,v_j)}{V_x(t,v_j)} = \left[\int_t^{\infty} e^{-\int_t^{\tau}[r(s)+\sigma_v n_v(s)]ds} e^{\int_t^{\tau} \frac{\lambda_v n_v(s) v(s)}{1-v(s)} ds} e^{\int_t^{\tau} g_Y(s)ds} d\tau \right]^{-1} = \Omega_t^{-1}$$

We therefore have a first restriction

$$\frac{\pi_x(t,\bar{v}_t)}{V_x(t,\bar{v}_t)} = \sigma_1 \alpha\beta L_t \frac{1-\bar{v}_t}{1-v_t} = \Omega_t^{-1} = \frac{\pi_x(t,v_j)}{V_x(t,v_j)}$$

Let us study the growth rate of the value of the firm with the leading-edge technology of date t. Taking logs and differentiating the expression obtained above ($V_x(t,\bar{v}_t) = (1-\bar{v}_t)\,\alpha\beta\,(1-\alpha)\,\frac{Y_t}{(1-v_t)}\Omega_t$):

$$\frac{\dot{V}_x(t,\bar{v}_t)}{V_x(t,\bar{v}_t)} = \frac{\dot{Y}_t}{Y_t} + \frac{\dot{v}_t}{(1-v_t)} + \frac{\dot{\Omega}_t}{\Omega_t}$$

While the value of the next innovation $V_x(t+dt,\bar{v}_{t+dt}) = (1-\bar{v}_{t+dt})\,\alpha\beta\,(1-\alpha)\,\frac{Y_{t+dt}}{(1-v_{t+dt})}\Omega_{t+dt}$. Consider now the expected growth rate of the value of innovations: the change of the value of the most valuable firm across the economy, the one owning the leading-edge technology. This firm will obviously change identity. We have:

$$\begin{aligned}
\left.\frac{\dot{V}_x}{V_x}\right|_{\text{innov}} &= \lim_{dt\to 0}\frac{V_x(t+dt,\bar{v}_{t+dt}) - V_x(t,\bar{v}_t)}{dt}\frac{1}{V_x(t,\bar{v}_t)} \\
\text{"} &= \lim_{dt\to 0}\left[\left(\frac{1-\bar{v}_{t+dt}}{1-\bar{v}_t}\right)\frac{Y_{t+dt}}{Y_t}\left(\frac{1-v_t}{1-v_{t+dt}}\right)\frac{\Omega_{t+dt}}{\Omega_t} - 1\right]\frac{1}{dt} \\
\text{"} &\simeq \frac{\lambda_v n_{vt}\bar{v}_t}{1-\bar{v}_t} + \frac{\dot{Y}_t}{Y_t} + \frac{\dot{v}_t}{(1-v_t)} + \frac{\dot{\Omega}_t}{\Omega_t} \\
\text{"} &= \frac{\lambda_v n_{vt}\bar{v}_t}{1-\bar{v}_t} + \frac{\dot{V}_x(t,\bar{v}_t)}{V_x(t,\bar{v}_t)}
\end{aligned}$$

This result can be useful because the arbitrage condition $\sigma_v V_x(s,\bar{v}_s) = (1-\alpha)\,\frac{Y_s}{L_s}$ holds for innovations at all dates s. That is we have that:

$$\left.\frac{\dot{V}_x}{V_x}\right|_{\text{innov}} = \frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t}$$

Instead the Bellman equations $\frac{\dot{V}_x}{V_x} + \frac{\pi_x}{V_x} = r + \sigma_v n_v$ describe the evolution of the value of a firm with a fixed technology v_j, for instance they apply to the innovator of date t. Hence:

$$\frac{\dot{V}_x(t,\bar{v}_t)}{V_x(t,\bar{v}_t)} = r + \sigma_v n_v - \frac{\pi_x}{V_x}$$

Bringing together these last three equations we have

$$\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{\lambda_v n_{vt}\bar{v}_t}{1-\bar{v}_t} + r_t + \sigma_v n_{vt} - \frac{\pi_{xt}}{V_{xt}}$$

Finally, substituting for $r_t = \rho + \varepsilon\frac{\dot{Y}_t}{Y_t}$ and $\frac{\pi_{xt}}{V_{xt}} = \Omega_t^{-1}$:

$$\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{\lambda_v n_{vt}\bar{v}_t}{1-\bar{v}_t} + \rho + \varepsilon\frac{\dot{Y}_t}{Y_t} + \sigma_v n_{vt} - \Omega_t^{-1}$$

if instead we use $\frac{\pi_{xt}}{V_{xt}} = \sigma_v\alpha\beta L_t\frac{1-\bar{v}_t}{1-v_t}$:

$$\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{\lambda_v n_{vt}\bar{v}_t}{1-\bar{v}_t} + \rho + \varepsilon\frac{\dot{Y}_t}{Y_t} + \sigma_v n_{vt} - \sigma_v\alpha\beta L_t\frac{1-\bar{v}_t}{1-v_t}$$

Taking logs of aggregate output $Y = \left[\left(\alpha^2\beta\right)^{\beta} \left(\alpha^2 (1-\beta)\right)^{1-\beta} \right]^{\frac{\alpha}{1-\alpha}} L (1-v)^{\beta} (1-u)^{1-\beta} \left(q_t^{\beta} c^{1-\beta}\right)^{\frac{-\alpha}{1-\alpha}}$, and differentiating :

$$\begin{aligned}
\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} &= -\beta \frac{\dot{v}_t}{1-v_t} - (1-\beta) \frac{\dot{u}_t}{1-u_t} - \frac{\alpha\beta}{1-\alpha} r_t \\
\text{''} &= -\beta \frac{\dot{v}_t}{1-v_t} - (1-\beta) \frac{\dot{u}_t}{1-u_t} - \frac{\alpha\beta}{1-\alpha} \left(\rho + \varepsilon \frac{\dot{Y}_t}{Y_t} \right) \\
\left(\frac{1-\alpha+\varepsilon\alpha\beta}{1-\alpha} \right) \frac{\dot{Y}_t}{Y_t} &= \frac{\dot{L}_t}{L_t} - \beta \frac{\dot{v}_t}{1-v_t} - (1-\beta) \frac{\dot{u}_t}{1-u_t} - \frac{\alpha\beta}{1-\alpha} \rho \\
\frac{\dot{Y}_t}{Y_t} &= M \left[\frac{\dot{L}_t}{L_t} - \beta \frac{\dot{v}_t}{1-v_t} - (1-\beta) \frac{\dot{u}_t}{1-u_t} - \frac{\alpha\beta}{1-\alpha} \rho \right]
\end{aligned}$$

where $M = \frac{1-\alpha}{1-\alpha+\varepsilon\alpha\beta}$. hence

$$(1-\varepsilon) \frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{\lambda_v n_{vt} \bar{v}_t}{1-\bar{v}_t} + \rho + \sigma_v n_{vt} - \sigma_v \alpha\beta L_t \frac{1-\bar{v}_t}{1-v_t}$$

$$\begin{aligned}
\left[(1-\varepsilon) M - 1 \right] \frac{\dot{L}_t}{L_t} &= (1-\varepsilon) M \left[+\beta \frac{\dot{v}_t}{1-v_t} + (1-\beta) \frac{\dot{u}_t}{1-u_t} + \frac{\alpha\beta}{1-\alpha} \rho \right] \\
&\quad + \frac{\lambda_v n_{vt} \bar{v}_t}{1-\bar{v}_t} + \rho + \sigma_v n_{vt} - \sigma_v \alpha\beta L_t \frac{1-\bar{v}_t}{1-v_t}
\end{aligned}$$

$$\begin{aligned}
\frac{\dot{L}_t}{L_t} &= \frac{(1-\varepsilon) M}{(1-\varepsilon) M - 1} \left[-\beta \frac{\lambda_v n_{vt} v_t}{1-v_t} - (1-\beta) \frac{\lambda_u n_{ut} u_t}{1-u_t} + \frac{\alpha\beta}{1-\alpha} \rho \right] \\
&\quad + \frac{1}{\left[(1-\varepsilon) M - 1 \right]} \left(\frac{\lambda_v n_{vt} \bar{v}_t}{1-\bar{v}_t} + \rho + \sigma_v n_{vt} - \sigma_v \alpha\beta L_t \frac{1-\bar{v}_t}{1-v_t} \right)
\end{aligned}$$

$$\begin{aligned}
\left[1 - (1-\varepsilon) M \right] \frac{\dot{n}_t}{L_t} &= \left[(1-\varepsilon) M \left(\frac{\lambda_u u_t}{1-u_t} - \beta \left[\frac{\lambda_v v_t}{1-v_t} + \frac{\lambda_u u_t}{1-u_t} \right] \right) + \left(\frac{\lambda_{vt} \bar{v}_t}{1-\bar{v}_t} + \sigma_v \right) \right] \phi n_t \\
&\quad - \left[(1-\varepsilon) M (1-\beta) \frac{\lambda_u u_t}{1-u_t} - \sigma_1 \alpha\beta n_t \frac{1-\bar{v}_t}{1-v_t} \right] n_t \\
&\quad + \left[(1-\varepsilon) M \frac{\alpha\beta}{1-\alpha} + 1 \right] \rho - \sigma_v \alpha\beta N \frac{1-\bar{v}_t}{1-v_t}
\end{aligned}$$

4 Conlusions

The timing and the relative size of R&D effort are typically different in the decentralized and the centralized economies.

First, the system does not admit any balanced path for the decentralized economy, because the growth rate in the efficiency of exploitation of any resource cannot be constant. Both effort in R&D sectors is decreasing since the marginal reward to R&D activity declines as the upper bound approaches. Hence, we pursue the analysis using numerical methods. Second, in the decentralized economy, R&D effort is decreasing in the resource-specific efficiency level, because the marginal reward to R&D activity declines as the upper bound approaches. As a consequence, the share of resources devoted to R&D falls asymptotically towards zero. Third, the finiteness of efficiency improvements together with that of the non-renewable resource supply, imply that the initial conditions determine the qualitative feature of the transition path. In this case, R&D firms make choices taking into account a limited time horizon since successive innovations make patents obsolete. R&D tends to concentrate in the resource sector where the demand is the largest (a scale effect that depends on the efficiency gap between and on the relative scarcity of the two resources) and where the scope for efficiency improvement is the highest (most important in the backward sector). When the first effect dominates, the technological gap increases, giving rise to possible unbalanced rushes of R&D on one resource. When the initial efficiency gap is favorable to the non-renewable resource, our model predicts that R&D activity focuses on this resource first, causing the gap to increase. Eventually, as the growth of the resource rent reduces the competitiveness of the non-renewable resource, R&D focuses gradually and eventually exclusively on the renewable substitute.

This development is in contrast with the qualitative features of the optimal path of technological change. Because of the limited availability of the non renewable resource, whenever efficiency improvement on this resource is worth, it is preferable from a social point of view to obtain it as soon as possible. In fact, the improved efficiency can be applied to a larger resource stock if it is obtained early rather than late. To sum up, we show the optimal path is divided into three phases: first an unbalanced path with a rise in the efficiency of one resource; second a balance path with a rise in both resources efficiency; and last the steady state.

5 Appendices

5.1 Appendix 1: Social Planner With Only Fossil Resource

This version of the problem without substitute: it takes utility as the objective function and is based on the crucial assumption of constant returns to scale in R&D. In fact, the marginal return on R&D must be bounded from above, otherwise as the resources employed in R&D fall, the marginal reward from R&D investment increases indefinitely. Thus R&D never stops and the economy converges toward the origin of the system ($v = u = n = 0$) without ever reaching it. The origin is not a well behaved steady state, because the Jacobian of the linearized system is not defined. In fact the region of convergence shrinks to an empty space. Assuming CRS in R&D allows us to derive a well defined steady state.

The social planner problem is:

$$\left\{ \begin{array}{l} \max\limits_{\{(n_t, x_t)\}_{t=0}^{+\infty}} \int\limits_0^{+\infty} e^{-\rho t} u\left(C_t\right) dt \\ \text{s.t.:} \left| \begin{array}{l} C_t = Y_t = \left[\left(1 - v_t\right)\left(N - n_t\right)\right]^{1-\alpha} x_t^{\alpha} \\ \dot{X}_t = -x_t \\ \dot{v}_t = -\lambda n_t v_t \end{array} \right. \end{array} \right. \tag{13}$$

then

$$\begin{aligned} H_c &= u\left(Y\right) + \kappa\left(-\dot{v}\right) + \mu \dot{X} \\ " &= u\left(\left[\left(1 - v_t\right)\left(N - n_t\right)\right]^{1-\alpha} x_t^{\alpha}\right) + \kappa_t \lambda n_t v_t - \mu_t x_t \end{aligned}$$

hence

$$\left\{ \begin{array}{l} \frac{\partial H}{\partial n} = -u'\left(.\right)\left(1-\alpha\right)\frac{Y}{(N-n)} + \kappa\lambda v = 0 \\ \frac{\partial H}{\partial x} = u'\left(.\right)\alpha\frac{Y}{x} - \mu = 0 \\ -\frac{\partial H}{\partial(-v)} = -u'\left(.\right)\left(1-\alpha\right)\frac{Y}{(1-v)} + \kappa\lambda n = \dot{\kappa} - \rho\kappa \\ -\frac{\partial H}{\partial X} = 0 = \dot{\mu} - \rho\mu \end{array} \right. \tag{14}$$

Using (25) to substitute for $u'\left(.\right)\left(1-\alpha\right)\frac{Y}{\kappa} = \lambda v\left(N-n\right)$, we eliminate the costate variable κ from (14):

$$\begin{aligned} \frac{\dot{\kappa}}{\kappa} &= \rho + \lambda n - u'\left(.\right)\left(1-\alpha\right)\frac{Y}{\kappa\left(1-v\right)} \\ " &= \rho + \lambda n - \lambda\left(N-n\right)\frac{v}{1-v} \end{aligned} \tag{15}$$

The ratio of (25) to (44) gives:

$$\frac{\alpha\left(N-n\right)}{\left(1-\alpha\right)x} = \frac{\mu}{\kappa\lambda v}$$

taking logs and differentiating:

$$\frac{-\dot{n}}{\left(N-n\right)} - \frac{\dot{x}}{x} = \frac{\dot{\mu}}{\mu} - \frac{\dot{\kappa}}{\kappa} - \frac{\dot{v}}{v}$$

which, when using (28) and (14), simplifies to

$$\frac{-\dot{n}}{\left(N-n\right)} - \frac{\dot{x}}{x} = \lambda\left(N-n\right)\frac{v}{1-v} \tag{16}$$

Furthermore, taking logs and differentiating (44) we have:

$$\frac{u''\dot{C}}{u'} + \frac{\dot{Y}}{Y} = \frac{\dot{\mu}}{\mu} + \frac{\dot{x}}{x}$$

Assuming constant intertemporal elasticity of substitution $\sigma = -u'/u''C$, taking into account that $\frac{\dot{C}}{C} = \frac{\dot{Y}}{Y}$ and using (14), the previous expression gives

$$\frac{\dot{x}}{x} = \left(1 - \frac{1}{\sigma}\right)\frac{\dot{Y}}{Y} - \rho \tag{17}$$

>From the production function we have that the growth rate of output is:

$$\begin{aligned}\frac{\dot{Y}}{Y} &= (1-\alpha)\left(\frac{\dot{v}}{1-v} + \frac{\dot{n}}{(N-n)}\right) + \alpha\frac{\dot{x}}{x} \\ \text{"} &= (1-\alpha)\left(\lambda n\frac{v}{1-v} - \frac{n}{(N-n)}\frac{\dot{n}}{n}\right) + \alpha\frac{\dot{x}}{x} \qquad (18)\end{aligned}$$

Therefore, substituting $\frac{\dot{x}}{x}$ from (17) into(16) and (18) we get the system:

$$\begin{aligned}(1-\alpha)\frac{n}{(N-n)}\frac{\dot{n}}{n} + \left[1-\alpha\left(1-\frac{1}{\sigma}\right)\right]\frac{\dot{Y}}{Y} &= (1-\alpha)\lambda n\frac{v}{1-v} - \alpha\rho \\ \frac{n}{(N-n)}\frac{\dot{n}}{n} + \left(1-\frac{1}{\sigma}\right)\frac{\dot{Y}}{Y} &= \rho - \lambda(N-n)\frac{v}{1-v}\end{aligned}$$

from which:

$$(1-\alpha)\frac{n}{(N-n)}\left(\frac{\dot{n}}{n}\right) = -\alpha\rho + (1-\alpha)\lambda n\frac{v}{1-v} - \left[1-\alpha\left(1-\frac{1}{\sigma}\right)\right]\left[\frac{\rho - \lambda(N-n)\frac{v}{1-v} - \frac{n}{(N-n)}\frac{\dot{n}}{n}}{1-\frac{1}{\sigma}}\right]$$

$$\frac{n}{(N-n)}\left(\frac{\dot{n}}{n}\right) = \sigma\rho + [n - (\sigma + \alpha(1-\sigma))N]\lambda\frac{v}{1-v}$$

$$\dot{n} = (N-n)\left\{\sigma\rho + [n - (\sigma + \alpha(1-\sigma))N]\lambda\frac{v}{1-v}\right\} \tag{19}$$

Plugging this result back into the system

$$\frac{\dot{Y}}{Y} = \sigma\left[(1-\alpha)\lambda N\frac{v}{1-v} - \rho\right] \tag{20}$$

so that (17) becomes:

$$\frac{\dot{x}}{x} = -\sigma\rho - (1-\sigma)(1-\alpha)\lambda N\frac{v}{1-v} \tag{21}$$

The system of differential equations (23)-(29) admits a stationary state in $(v^*, 0)$ since

$$v^* = \frac{1}{1 + m\frac{N\lambda}{\sigma\rho}} \tag{22}$$

where we define $m = \sigma + \alpha(1-\sigma) = (1-\alpha)\sigma + \alpha > 0$. The steady state is characterized by:

$$\begin{aligned} \frac{\dot{Y}}{Y} &\rightarrow -\frac{\alpha\sigma\rho}{m} \\ \frac{\dot{x}}{x} &\rightarrow \frac{-\sigma\rho}{m} \end{aligned}$$

The phase diagram is described by the locus $\dot{v} = 0$ which coincides with the v-axis, and $\dot{n} = 0$ for $n = N$ or

$$\begin{aligned}
n &= h(v) = mN - \frac{\sigma\rho}{\lambda}\frac{1-v}{v} \\
\dot{n} &> 0 \ \ \forall n > h(v) \quad \text{and} \quad \dot{n} < 0 \ \ \forall n < h(v) \\
h' &= \frac{\sigma\rho}{\lambda}\frac{1}{v^2} > 0 \qquad h'' = -\frac{\sigma\rho}{2\lambda}\frac{1}{v^3} < 0 \qquad \lim_{v\to 0^+} h(v) = -\infty \\
N &= h(\bar{v}) : \ \ \bar{v} = \frac{1}{1-(1-\alpha)(1-\sigma)\frac{\lambda N}{\sigma\rho}} \\
\sigma &< 1 \Rightarrow \bar{v} > 1 \ ; \quad \sigma \in \left(\left[1+\frac{\rho}{(1-\alpha)\lambda N}\right]^{-1}, 1\right) \Rightarrow \bar{v} \in (0,1) \\
\sigma &< \left[1+\frac{\rho}{(1-\alpha)\lambda N}\right]^{-1} \Rightarrow \bar{v} < 0
\end{aligned}$$

The Jacobian of the system $[\dot{n};\dot{v}]$:

$$\begin{vmatrix} -\sigma\rho + [1+m]\lambda N\frac{v^*}{1-v^*} & -m\frac{\lambda N^2}{(1-v^*)^2} \\ -\lambda v^* & 0 \end{vmatrix}$$

The linearized system is

$$\begin{vmatrix} \dot{n} \\ \dot{v} \end{vmatrix} = \begin{vmatrix} \frac{\rho\sigma}{m} & -\frac{[\sigma\rho+m\lambda N]^2}{m} \\ -\lambda\left[1+m\frac{N\lambda}{\sigma\rho}\right]^{-1} & 0 \end{vmatrix}\begin{vmatrix} n \\ v-v^* \end{vmatrix}$$

hence $\Delta = -m(\lambda N)^2 v^*/(1-v^*)^2 < 0$, $Tr = \rho\sigma/m$, implying that the stationary state is a saddle point.

5.2 Appendix 2: Social Planner, *The Wrong Analysis* (!)[4]

5.2.1 Social Planner: No Renewable Substitute

The social planner takes into account the technological constraints (53) and (69), and maximizes output subject to the labor, resource and efficiency accumulation

[4] An unfruitfull but stimulating analysis.

constraints:

$$\left\{\begin{array}{l} \max\limits_{\left\{n_t,\{x_{jt}\}_{j=0}^{1}\right\}_{t=0}^{\infty}} Y_t = (N-n_t)^{1-\alpha} \int\limits_0^1 (1-v_{jt}) \left(\frac{x_{jt}}{1-v_{jt}}\right)^{\alpha} dj \\ \text{s.t.:} \left| \begin{array}{l} \dot{X}_t = -\int\limits_0^1 x_{jt} dj \qquad\qquad \forall t,\ X,\ x \geq 0 \\ \dot{v}_t = -\lambda n_t^{\delta} v_t \end{array} \right. \end{array}\right. \tag{23}$$

Notice that we have changed the law of motion of waste by introducing decreasing returns to R&D ($\delta \in (0,1)$) and that we are assuming that the social planner controls directly the evolution of the average waste (and thus of average efficiency) by adjusting appropriately R&D employment.[5]

First we consider the cross-sectoral allocation of the resource when its value is denoted by μ. he f.o.c. is $\forall j \in [0,1]$:

$$\alpha (N-n_t)^{1-\alpha} (1-v_{jt})^{1-\alpha} x_{jt}^{\alpha-1} = \mu_t$$

thus $\forall j$

$$\tilde{x}_t = \frac{x_{jt}}{1-v_{jt}} = \alpha^{\frac{1}{1-\alpha}} \mu_t^{\frac{-1}{1-\alpha}} (N-n_t) \tag{24}$$

Let us use directly this optimal allocation to write the current value *Hamiltonian* of the problem:

$$\begin{aligned} H_c &= Y_t - \kappa_t \dot{v}_t + \mu_t \dot{X}_t \\ \text{"} &= (N-n_t)^{1-\alpha} \int_0^1 (1-v_{jt}) \alpha^{\frac{\alpha}{1-\alpha}} \mu_t^{\frac{-\alpha}{1-\alpha}} (N-n_t)^{\alpha} dj + \kappa_t \lambda n_t^{\delta} v_t \\ & \quad -\mu_t \int_0^1 (1-v_{jt}) \alpha^{\frac{1}{1-\alpha}} \mu_t^{\frac{-1}{1-\alpha}} (N-n_t) dj \\ \text{"} &= \alpha^{\frac{\alpha}{1-\alpha}} \mu_t^{\frac{-\alpha}{1-\alpha}} (N-n_t) \int_0^1 (1-v_{jt}) dj + \kappa_t \lambda n_t^{\delta} v_t - \alpha^{\frac{1}{1-\alpha}} \mu_t^{\frac{-\alpha}{1-\alpha}} (N-n_t) \int_0^1 (1-v_{jt}) dj \\ \text{"} &= \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha) \mu_t^{\frac{-\alpha}{1-\alpha}} (N-n_t) \left(1 - \int_0^1 v_{jt} dj\right) + \kappa_t \lambda n_t^{\delta} v_t \\ \text{"} &= A \mu_t^{\frac{-\alpha}{1-\alpha}} (N-n_t)(1-v_t) + \kappa_t \lambda n_t^{\delta} v_t \end{aligned}$$

defining $A = \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha)$. Let ρ be the social discount rate. The following are the f.o.c. w.r.t. n, and the Euler conditions for X and $(-v)$ (if $\kappa \geq 0$):

$$A \mu_t^{\frac{-\alpha}{1-\alpha}} (1-v_t) = \delta \kappa_t \lambda n_t^{\delta-1} v_t \tag{25}$$

[5] That is, for the moment we abstract from the fact that the simple relation between efficiency growth and n exists for the leading edge technology and not for the average one. In fact, the average index is computed taking into account the distribution of sales across sectors j's, which is controlled here by the social planner.

$$\dot{\mu}_t = \rho\mu_t \qquad \Rightarrow \qquad \mu_t = \mu_0 e^{\rho t} \tag{26}$$

$$\dot{\kappa}_t = \rho\kappa_t - A\mu_t^{\frac{-\alpha}{1-\alpha}}(N - n_t) + \kappa_t\lambda n_t^{\delta} \tag{27}$$

Using (25) we can rewrite (27) as:

$$\frac{\dot{\kappa}_t}{\kappa_t} = \rho + \lambda n_t^{\delta} - \delta\lambda n_t^{\delta}\left(\frac{N}{n_t} - 1\right)\frac{v_t}{1 - v_t} \tag{28}$$

Logdifferentiating (25):

$$\frac{-\alpha}{1-\alpha}\rho = \frac{\dot{\kappa}_t}{\kappa_t} + (\delta - 1)\frac{\dot{n}_t}{n_t} + \frac{\dot{v}_t}{v_t} + \frac{\dot{v}_t}{1 - v_t}$$

which we can rearrange using (23) and (28) to obtain:

$$\begin{aligned}
\frac{\dot{n}_t}{n_t}(1-\delta) &= \frac{\alpha}{1-\alpha}\rho + \rho + \lambda n_t^{\delta} - \delta\lambda n_t^{\delta}\left(\frac{N}{n_t} - 1\right)\frac{v_t}{1-v_t} - \lambda n_t^{\delta} - \frac{\lambda n_t^{\delta} v_t}{1-v_t} \\
\text{''} &= \frac{1}{1-\alpha}\rho - \delta\lambda n_t^{\delta}\left(\frac{N}{n_t} - 1\right)\frac{v_t}{1-v_t} - \lambda n_t^{\delta}\frac{v_t}{1-v_t}
\end{aligned}$$

and finally:

$$\frac{\dot{n}_t}{n_t} = \frac{\rho}{(1-\alpha)(1-\delta)} - \lambda n_t^{\delta}\frac{v_t}{1-v_t}\left[1 + \frac{\delta}{1-\delta}\frac{N}{n_t}\right] \tag{29}$$

We have a system of two differential equations (23)-(29) that rules the path of R&D effort and therefore of the resource efficiency.

Equation (29) can also be read as

$$\dot{n}_t = \frac{\rho n_t}{(1-\alpha)(1-\delta)} - \lambda n_t^{1+\delta}\frac{v_t}{1-v_t}\left[1 + \frac{\delta}{1-\delta}\frac{N}{n_t}\right]$$

By setting $\dot{n} = 0$, we define the function:

$$v = f(n) = \left[1 + (1-\alpha)(1-\delta)\frac{\lambda}{\rho}n^{\delta}\left(1 + \frac{\delta}{1-\delta}\frac{N}{n}\right)\right]^{-1}$$

with

$$\begin{aligned}
f'(n) &= \frac{(1-\alpha)(1-\delta)\frac{\lambda}{\rho}\delta n^{\delta-2}(N-n)}{\left[1 + (1-\alpha)(1-\delta)\frac{\lambda}{\rho}n^{\delta}\left(1 + \frac{\delta}{1-\delta}\frac{N}{n}\right)\right]^2} > 0 \\
\bar{v} &\equiv \lim_{n \to N} f(n) = \left[1 + (1-\alpha)\frac{\lambda}{\rho}N^{\delta}\right]^{-1}
\end{aligned}$$

Numerical simulations show that $f(n)$ is concave and has an infinite slope at $n = 0$. Hence the inverse of f defines the locus of $\dot{n} = 0$ in the (v, n) phase plane as a convex curve. Above the curve $\dot{n} > 0$ and below $\dot{n} < 0$. From the phase diagram we see that there exists a unique optimal path converging towards the origin. Along this path, R&D employment falls steadily approaching zero only asymptotically. Therefore, waste, v, decreases indefinitely and efficiency is continuously improved.

Proof of the multiplicity of solutions.

Let us study the trajectories that are locally optimal within the region of interest defined by the space below the $\dot{n} = 0$ locus and inside the feasible region $n \in [0, N]$ and $v \in (0, 1)$.

Define $A = \rho / (1 - \alpha)(1 - \delta)$.

First we analyze the slope of the locus $\dot{n} = 0$ in the (v, n) space. A part for the solution $n = 0$ the locus is defined by the implicit function:

$$\begin{aligned}
F(v, n) &= A - \lambda n^{\delta - 1} \frac{v}{1 - v} \left(n + \frac{\delta}{1 - \delta} N \right) = 0 \\
\frac{dF}{dn} &= \lambda n^{\delta - 2} \frac{v}{1 - v} \delta (N - n) > 0 \\
\frac{dF}{dv} &= - \frac{\lambda n^{\delta - 1}}{(1 - v)^2} \left(n + \frac{\delta}{1 - \delta} N \right) < 0 \\
\left. \frac{dn}{dv} \right|_{\dot{n} = 0} &= \frac{n \left(n + \frac{\delta}{1 - \delta} N \right)}{\delta (N - n) v (1 - v)} > 0
\end{aligned}$$

for $n = N$ we have $\dot{n} = 0$ only if $v = \bar{v} = \left[1 + (1 - \alpha) \frac{\lambda}{\rho} N^{\delta} \right]^{-1}$. Moreover $\lim_{n \to 0^+} \left. \frac{dn}{dv} \right|_{\dot{n} = 0} = 0$, $\lim_{n \to N^-} \left. \frac{dn}{dv} \right|_{\dot{n} = 0} = \infty$.

All the trajectories $\hat{n}(v)$ satisfying the local optimality conditions are characterized by a slope:

$$\left. \frac{dn}{dv} \right|_{\hat{n}(v)} = \frac{\dot{n}}{\dot{v}} = - \frac{A}{\lambda} \frac{n^{1 - \delta}}{v} + \frac{1}{1 - v} \left(n + \frac{\delta}{1 - \delta} N \right)$$

implying the following properties:

1. Define the function:

$$\begin{aligned}
\tilde{v}(n) &= \frac{A n^{1 - \delta} / \lambda}{A n^{1 - \delta} / \lambda + n + \frac{\delta}{1 - \delta} N} \text{ such that } \left. \frac{dn}{dv} \right|_{\hat{n}(v)} = 0 \\
\forall v &> \tilde{v} \left. \frac{dn}{dv} \right|_{\hat{n}(v)} > 0 \text{ and } \forall v < \tilde{v} \left. \frac{dn}{dv} \right|_{\hat{n}(v)} < 0
\end{aligned}$$

since $d^2 n / dv^2 = A n^{1 - \delta} / v^2 + (n + \frac{\delta}{1 - \delta} N) / \left(1 - v^2 \right) > 0$. Notice that $\tilde{v}(0) = 0$, so that at the origin the slope of the optimal policy is zero. In other words, all trajectories have their minimum at $\tilde{v}(n)$ and only one of this trajectories has its minimum at the origin, $v = 0$ and $n = 0$.

2. $\forall v \in (0, \bar{v})$
$$\frac{d}{dn}\left(\left.\frac{dn}{dv}\right|_{\hat{n}(v)}\right) < 0$$
In fact: $\frac{d}{dn}\left(\left.\frac{dn}{dv}\right|_{\hat{n}(v)}\right) = g(v,n) = 1/(1-v) - (1-\delta) A/\left(\lambda v n^{\delta}\right) \Rightarrow \exists$ locus $g(v,n) = 0$ in the (v,n) space $[\tilde{n}(v) = [(1-v)(1-\delta) A/\lambda v]^{1/\delta}]$ such that $\frac{d}{dn}\left(\left.\frac{dn}{dv}\right|_{\hat{n}(v)}\right) < 0$ below this locus. The $g = 0$ locus is characterized by $\left.\frac{dn}{dv}\right|_{g=0} < 0$, and $g(\bar{v}, n) = 0$ implies $n = N \Rightarrow$ the region $v \in (0, \bar{v})$ and $n \in (0, N)$ is entirely below the $g = 0$ locus.

3. For $n = 0$:
$$\frac{d}{dv}\left(\left.\frac{dn}{dv}\right|_{\hat{n}(v)}\right) > 0$$
since $\lim_{n \to 0} \left.\frac{dn}{dv}\right|_{\hat{n}(v)} = \frac{\delta}{1-\delta}\frac{N}{1-v} > 0$ which is increasing in v.

4. $\forall n > 0$ the slope of the $\dot{n} = 0$ locus is positive if it crosses the policy function $\hat{n}(v)$ (the latter is flat by definition on the $\dot{n} = 0$ locus). In fact, substituting for $\tilde{v}(n)$ into $\left.\frac{dn}{dv}\right|_{\dot{n}=0}$ we get:
$$\left.\frac{dn}{dv}\right|_{\dot{n}=0} = \frac{n^{\delta}\left[An^{\delta-1} + \lambda\left(n + \frac{\delta}{1-\delta}N\right)\right]^2}{\lambda A \delta (N-n)} > 0$$

Consider $v_0 < \bar{v}$. Define our candidate solution as the optimal policy $n^*(v)$ starting at (v_0, n_0^*) and converging asymptotically towards the origin. From 1 we know that it is a increasing function with a minimum at the origin, i.e. a zero slope asymptotically.

All policies starting with $n > n_0^*$ are described by a flatter schedule than $n^*(v)$ (property2). The trajectory will cross the $\dot{n} = 0$ schedule in finite time (property 4). Thereafter $\dot{n} > 0$ forever and the economy diverges from the origin. (property 1). Asymptotically, either the trajectory converges towards $n = N$ with $v > 0$ or it first hits the $v = 0$ axes and then moves upwards along it until $n = N$ (since $\dot{n} > 0$). Both results are not optimal.

All policies starting with $n < n_0^*$ are described by a steeper schedule than $n^*(v)$ (property2). They attain the $n = 0$ axes in finite time and there the system rests (remember that $\dot{n} = 0$ along the $n = 0$ axes).

To discriminate between the policies starting with $n < n_0^*$ and our candidate $n^*(v)$, we have to use the transversality condition (using (28)):
$$\begin{aligned}
\lim_{t\to\infty} e^{-\rho t}\kappa_t v_t &= e^{\left(\frac{\dot{\kappa}}{\kappa} + \frac{\dot{v}}{v} - \rho\right)t}\kappa_0 v_0 \\
" &= \kappa_0 v_0 e^{\int_0^t \left(\rho - \frac{\dot{v}_s}{v_s} - \delta\lambda n_s^{\delta-1}(N-n_s)\frac{v_s}{1-v_s} + \frac{\dot{v}_s}{v_s} - \rho\right) ds} \\
" &= \kappa_0 v_0 e^{-\delta\lambda \int_0^t n_s^{\delta-1}(N-n_s)\frac{v_s}{1-v_s} ds} = 0
\end{aligned}$$

which is satisfied if and only if

$$\lim_{t\to\infty}\int_0^t n_s^{\delta-1}\left(N-n_s\right)\frac{v_s}{1-v_s}ds=\infty$$

that is when the integrand is unbounded. Consider our cases:

1. If $v_\infty=\hat{v}>0$:
   $i)$ and if $n_\infty=0$: $\lim\limits_{t\to\infty} n_s^{\delta-1}\left(N-n_s\right)\frac{v_s}{1-v_s}=\frac{(N-0)}{0^{1-\delta}}\frac{\hat{v}}{1-\hat{v}}=\infty$
   $ii)$ or if $n_\infty=\hat{n}>0$: $\lim\limits_{t\to\infty} n_s^{\delta-1}\left(N-n_s\right)\frac{v_s}{1-v_s}=\frac{(N-\hat{n})}{\hat{n}^{1-\delta}}\frac{\hat{v}}{1-\hat{v}}>0$; but $n_\infty>0\Rightarrow\dot{v}_\infty<0$ in contradiction with $v_\infty=\hat{v}>0$.

2. If $v_\infty=0$:
   $i)$ and if $n_\infty=0$: $\lim\limits_{t\to\infty} n_s^{\delta-1}\left(N-n_s\right)\frac{v_s}{1-v_s}=\frac{(N-0)}{0^{1-\delta}}\frac{0}{1-0}=N0^{\delta}=0$ (assuming that n and v converge to 0 at the same rate) $\Rightarrow$ the transversality condition is not satisfied
   $ii)$ or if $n_\infty=\hat{n}>0$: $\lim\limits_{t\to\infty} n_s^{\delta-1}\left(N-n_s\right)\frac{v_s}{1-v_s}=\frac{(N-\hat{n})}{\hat{n}^{1-\delta}}\frac{0}{1-0}=0\Rightarrow$ the transversality condition is not satisfied.

For all policies starting with $n<n_0^*$ we get $v_\infty=\hat{v}>0$

We conclude that according to $1i$ all policies leading to the stationary states defined by $n=0$ are optimal. Hence our candidate solution $n^*(v)$ is not unique. Furthermore if $2i$ is exact, then our candidate solution is not even optimal.

Open issues:
. - Is $2i$ correct?
. - What does it happen in the region $v\in(\bar{v},1)$ and $n\in(0,N)$?

5.2.2 Social Planner: Introducing a Renewable Substitute

Let us consider the possibility of substituting the fossil fuel primary input for a renewable one, denoted by s, available according to an infinitely elastic supply schedule at a constant unit cost c (in terms of output). Each resource input is transformed in the final output production function according to a resource-specific technology. The productivity of the intermediate good produced from the renewable resource, e_s, is denoted by $(1-u)$, where u measures the waste of the potential energy in the unit of exploited resource. Symmetrically, for $(1-v)$ is the efficiency index for the intermediate input produced using the fossil resource, e_x. The waste in each specific technology is reduced by resource-specific R&D activity. The latter employs labor according to:

$$\begin{aligned}\dot{v}_t &= -\lambda n_1^{\delta}v_t\\ \dot{u}_t &= -\lambda n_2^{\delta}u_t\end{aligned}$$

The two intermediate goods are imperfect substitutes in the aggregate final sector production function. Consider for instance a CES technology:

$$\begin{aligned} E_t &= \left[\beta e_x^{\frac{\sigma-1}{\sigma}} + (1-\beta)\, e_s^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\ I_t &= \left[\beta\,(1-v_t)^{\frac{\sigma-1}{\sigma}} + (1-\beta)\,(1-u_t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\ Y_t &= L_t^{1-\alpha} I_t E_t^{\alpha} \end{aligned}$$

Notice that with this specification, as the elasticity of substitution between resources tends to unity the production function converges to the following two stage Cobb-Douglas production function:

$$\begin{aligned} Y_t &= L_t^{1-\alpha}\left[(1-v_t)^{\beta}\,(1-u_t)^{1-\beta}\right]\left[e_x^{\beta} e_s^{1-\beta}\right]^{\alpha} \\ " &= L_t^{1-\alpha}\left[(1-v_t)\, e_x^{\alpha}\right]^{\beta}\left[(1-u_t)\, e_s^{\alpha}\right]^{1-\beta} \\ " &= L_t^{1-\alpha}\left\{\left[(1-v_t)^{\frac{1}{\alpha}}\, e_x\right]^{\beta}\left[(1-u_t)^{\frac{1}{\alpha}}\, e_s\right]^{1-\beta}\right\}^{\alpha} \end{aligned}$$

As a result, when the share of the renewable goes to zero ($\beta \to 1$), the model reduces to the one in the previous sections, since $Y_t = L_t^{1-\alpha}\,(1-v_t)\, e_x^{\alpha}$.

As in the last two sections we let the primary resource intensity of intermediate goods be proportional to their efficiency index, i.e.:

$$e_x = \frac{x}{1-v} \qquad \text{and} \qquad e_s = \frac{s}{1-u}$$

The problem has 4 controls (x, s, n_1, n_2), 3 states, X, v, and u, with 3 (positive) costate variables, μ, κ, and ζ respectively. Taking into account that labor is available i fixed supply N, the current value Hamiltonian is:

$$\begin{aligned} H &= Y_t - cs_t + \mu_t \dot{X}_t - \kappa_t \dot{v}_t - \zeta_t \dot{u}_t \\ " &= (N - n_{1t} - n_{2t})^{1-\alpha}\left[(1-v_t)^{\beta}\,(1-u_t)^{1-\beta}\right]\left[\left(\frac{x_t}{1-v_t}\right)^{\beta}\left(\frac{s_t}{1-u_t}\right)^{1-\beta}\right]^{\alpha} \\ &\quad - cs_t - \mu_t x_t + \kappa_t \lambda n_{1t}^{\delta} v_t + \zeta_t \lambda n_{2t}^{\delta} u_t \\ " &= \left[(N - n_{1t} - n_{2t})\,(1-v_t)^{\beta}\,(1-u_t)^{1-\beta}\right]^{1-\alpha}\left[x_t^{\beta} s_t^{1-\beta}\right]^{\alpha} - cs_t \\ &\quad - \mu_t x_t + \kappa_t \lambda n_{1t}^{\delta} v_t + \zeta_t \lambda n_{2t}^{\delta} u_t \end{aligned}$$

The following are the first order condition with respect to x, s, n_1 and n_2

respectively:

$$\mu_t = \alpha\beta\left[(N - n_{1t} - n_{2t})(1-v_t)^{\beta}(1-u_t)^{1-\beta}\right]^{1-\alpha} x_t^{\alpha\beta-1} s_t^{\alpha(1-\beta)} \tag{30}$$

$$c = \alpha(1-\beta)\left[(N - n_{1t} - n_{2t})(1-v_t)^{\beta}(1-u_t)^{1-\beta}\right]^{1-\alpha} x_t^{\alpha\beta-1} s_t^{\alpha(1-\beta)-1} \tag{31}$$

$$\delta\kappa_t\lambda n_{1t}^{\delta-1}v_t = (1-\alpha)\left[(1-v_t)^{\beta}(1-u_t)^{1-\beta}\right]^{1-\alpha}\left[\frac{x_t^{\beta}s_t^{1-\beta}}{N-n_{1t}-n_{2t}}\right]^{\alpha} \tag{32}$$

$$\delta\zeta_t\lambda n_{2t}^{\delta-1}u_t = (1-\alpha)\left[(1-v_t)^{\beta}(1-u_t)^{1-\beta}\right]^{1-\alpha}\left[\frac{x_t^{\beta}s_t^{1-\beta}}{N-n_{1t}-n_{2t}}\right]^{\alpha} \tag{33}$$

The first two conditions on resource allocation can be rewritten as $\alpha\beta Y/x = \mu$ and $\alpha(1-\beta)Y/s = c$, which together give:

$$x_t = \frac{\beta}{1-\beta}c\frac{s_t}{\mu_t} \tag{34}$$

The last two conditions on R&D effort equate the marginal product of labor in the final sector its marginal return in the two R&D sectors. Therefore equating the latter, we obtain the following arbitrage across R&D sectors:

$$\frac{n_{1t}}{n_{2t}} = \left(\frac{\kappa_t v_t}{\zeta_t u_t}\right)^{\frac{1}{1-\delta}} \tag{35}$$

The Euler condition with respect to the 3 states X, $(-v)$, and $(-u)$, respectively, are:

$$\dot{\mu}_t = \rho\mu_t \quad \Rightarrow \quad \mu_t = \mu_0 e^{\rho t} \tag{36}$$

$$\frac{\dot{\kappa}_t}{\kappa_t} = \rho + \lambda n_{1t}^{\delta} - \beta(1-\alpha)\frac{Y_t}{\kappa_t(1-v_t)} \tag{37}$$

$$\frac{\dot{\zeta}_t}{\zeta_t} = \rho + \lambda n_{2t}^{\delta} - (1-\beta)(1-\alpha)\frac{Y_t}{\zeta_t(1-u_t)} \tag{38}$$

Logdifferentiating (35) and substituting using (37) and (38):

$$\begin{aligned} \frac{d(n_1/n_2)\,dt}{n_1/n_2} &= \frac{1}{1-\delta}\left[\frac{\dot{\kappa}_t}{\kappa_t} - \frac{\dot{\zeta}_t}{\zeta_t} + \frac{\dot{v}_t}{v_t} - \frac{\dot{u}_t}{u_t}\right] \\ \frac{\dot{n}_{1t}}{n_{1t}} - \frac{\dot{n}_{2t}}{n_{2t}} &= \frac{1-\alpha}{1-\delta}Y_t\left[\frac{1-\beta}{\zeta_t(1-u_t)} - \frac{\beta}{\kappa_t(1-v_t)}\right] \end{aligned} \tag{39}$$

hence

$$\frac{d(n_1/n_2)\,dt}{n_1/n_2} >,=,< 0 \quad \Leftrightarrow \quad \frac{\kappa_t v_t}{\zeta_t u_t} \nearrow, \text{cst}, \searrow \quad \Leftrightarrow \quad \frac{\kappa_t(1-v_t)}{\zeta_t(1-u_t)} >,=,< \frac{\beta}{1-\beta}$$

>From (31) we have that Y/s is constant, i.e. $\dot{Y}/Y = \dot{s}/s$. Logdifferentiating the production function as written in H, we obtain:

$$\frac{\dot{Y}}{Y} = (1-\alpha)\left[-\frac{\dot{n}_1+\dot{n}_2}{(N-n_1-n_2)} - \beta\frac{\dot{v}}{1-v} - (1-\beta)\frac{\dot{u}}{1-u}\right] + \alpha\left[\beta\frac{\dot{x}}{x} + (1-\beta)\frac{\dot{s}}{s}\right]$$

Logdifferentiating (34) and using (36):

$$\frac{\dot{x}}{x} = \frac{\dot{s}}{s} - \frac{\dot{\mu}}{\mu} = \frac{\dot{s}}{s} - \rho$$

substituting in the previous expression:

$$\frac{\dot{Y}}{Y} = (1-\alpha)\left[-\frac{\dot{n}_1+\dot{n}_2}{(N-n_1-n_2)} - \beta\frac{\dot{v}}{1-v} - (1-\beta)\frac{\dot{u}}{1-u}\right] - \alpha\beta\rho + \alpha\frac{\dot{s}}{s}$$

and taking into account that $\dot{Y}/Y = \dot{s}/s$:

$$\begin{aligned} \left(\frac{\dot{Y}}{Y}=\right)\frac{\dot{s}}{s} &= -\frac{\dot{n}_1+\dot{n}_2}{(N-n_1-n_2)} - \beta\frac{\dot{v}}{1-v} - (1-\beta)\frac{\dot{u}}{1-u} - \frac{\alpha\beta}{1-\alpha}\rho && (40) \\ \text{''} &= \underbrace{\lambda\left(\beta n_1^{\delta}\frac{v}{1-v} + (1-\beta)\,n_2^{\delta}\frac{u}{1-u}\right)}_{\text{efficiency of energy inputs}} \underbrace{-\frac{\dot{n}_1+\dot{n}_2}{(N-n_1-n_2)}}_{\text{labor input}} \underbrace{-\frac{\alpha\beta}{1-\alpha}\rho}_{\text{fossil fuel input}} \end{aligned}$$

notice that if the asymptotic behavior of the system is such that $v, u \to 1$ and n_1, $n_2 \to 0$, then $\dot{Y}/Y \to -\frac{\alpha\beta}{1-\alpha}\rho \leq 0$, with equality only if output is inelastic w.r.t. fossil fuel inputs ($\beta = 0$).

Next, logdifferentiating (32) which can be written as $(1-\alpha)\,Y/\,(N-n_1-n_2) = \delta\kappa\lambda n_1^{\delta-1}v$:

$$\frac{\dot{Y}}{Y} + \frac{\dot{n}_1+\dot{n}_2}{(N-n_1-n_2)} = \frac{\dot{\kappa}}{\kappa} + \frac{\dot{v}}{v} - (1-\delta)\frac{\dot{n}_1}{n_1}$$

substituting for (37)

$$\frac{\dot{Y}}{Y} + \frac{\dot{n}_1+\dot{n}_2}{(N-n_1-n_2)} = \rho - \beta(1-\alpha)\frac{Y}{\kappa(1-v)} - (1-\delta)\frac{\dot{n}_1}{n_1}$$

Similarly, using (33) we get:

$$\frac{\dot{Y}}{Y} + \frac{\dot{n}_1+\dot{n}_2}{(N-n_1-n_2)} = \rho - (1-\beta)(1-\alpha)\frac{Y}{\zeta(1-u)} - (1-\delta)\frac{\dot{n}_2}{n_2}$$

combining these results we have:

$$\begin{aligned} (1-\delta)\frac{\dot{n}_1}{n_1} &= \frac{1-\alpha(1-\beta)}{(1-\alpha)}\rho - \beta(1-\alpha)\frac{Y}{\kappa(1-v)} - \lambda\left(\beta n_1^{\delta}\frac{v}{1-v} + (1-\beta)\,n_2^{\delta}\frac{u}{1-u}\right) \\ (1-\delta)\frac{\dot{n}_2}{n_2} &= \frac{1-\alpha(1-\beta)}{(1-\alpha)}\rho - (1-\beta)(1-\alpha)\frac{Y}{\zeta(1-u)} - \lambda\left(\beta n_1^{\delta}\frac{v}{1-v} + (1-\beta)\,n_2^{\delta}\frac{u}{1-u}\right) \end{aligned}$$

notice that substracting $\frac{\dot{n}_2}{n_2}$ from $\frac{\dot{n}_1}{n_1}$ we obtain again (9).

Using (32) and (33) to substitute for $(1-\alpha)\,Y/\kappa = \delta\lambda n_1^{\delta-1} v\,(N-n_1-n_2)$ and $(1-\alpha)\,Y/\zeta = \delta\lambda n_2^{\delta-1} u\,(N-n_1-n_2)$:

$$\begin{aligned}
(1-\delta)\frac{\dot{n}_1}{n_1} &= \frac{1-\alpha\,(1-\beta)}{(1-\alpha)}\rho - \beta\frac{v}{(1-v)}\delta\lambda n_1^{\delta-1}\,(N-n_1-n_2) \\
&\quad -\lambda\left(\beta n_1^{\delta}\frac{v}{1-v} + (1-\beta)\,n_2^{\delta}\frac{u}{1-u}\right) \\
(1-\delta)\frac{\dot{n}_2}{n_2} &= \frac{1-\alpha\,(1-\beta)}{(1-\alpha)}\rho - (1-\beta)\,\frac{u}{(1-u)}\delta\lambda n_2^{\delta-1}\,(N-n_1-n_2) \\
&\quad -\lambda\left(\beta n_1^{\delta}\frac{v}{1-v} + (1-\beta)\,n_2^{\delta}\frac{u}{1-u}\right)
\end{aligned}$$

hence:

$$(1-\delta)\left[\frac{\dot{n}_1}{n_1} - \frac{\dot{n}_2}{n_2}\right] = \delta\lambda\,(N-n_1-n_2)\left[(1-\beta)\,\frac{u}{(1-u)}n_2^{\delta-1} - \beta\frac{v}{(1-v)}n_1^{\delta-1}\right] \tag{41}$$

and the dynamic system is described by the following 4 differential equations:

$$\begin{aligned}
\dot{n}_1 &= \frac{1-\alpha\,(1-\beta)}{(1-\delta)\,(1-\alpha)}\rho n_1 - \beta\frac{v}{(1-v)}\frac{\delta\lambda n_1^{\delta}}{(1-\delta)}\,(N-n_1-n_2) \\
&\quad -\frac{\lambda n_1}{(1-\delta)}\left(\beta n_1^{\delta}\frac{v}{1-v} + (1-\beta)\,n_2^{\delta}\frac{u}{1-u}\right) \\
\dot{n}_2 &= \frac{1-\alpha\,(1-\beta)}{(1-\delta)\,(1-\alpha)}\rho n_2 - (1-\beta)\,\frac{u}{(1-u)}\frac{\lambda n_2^{\delta}}{(1-\delta)}\,(N-n_1-n_2) \\
&\quad -\frac{\lambda n_2}{(1-\delta)}\left(\beta n_1^{\delta}\frac{v}{1-v} + (1-\beta)\,n_2^{\delta}\frac{u}{1-u}\right) \\
\dot{v} &= -\lambda n_1^{\delta} v \\
\dot{u} &= -\lambda n_2^{\delta} u
\end{aligned}$$

or

$$\begin{aligned}
\dot{n}_1 &= \frac{1-\alpha\,(1-\beta)}{(1-\delta)\,(1-\alpha)}\rho n_1 - \beta\frac{v}{(1-v)}\frac{\delta}{(1-\delta)}\lambda n_1^{\delta}\,(N-n_2) \\
&\quad -\lambda n_1\left(\beta\frac{v}{1-v}n_1^{\delta} + (1-\beta)\,\frac{u}{1-u}\frac{n_2^{\delta}}{(1-\delta)}\right) \\
\dot{n}_2 &= \frac{1-\alpha\,(1-\beta)}{(1-\delta)\,(1-\alpha)}\rho n_2 - (1-\beta)\,\frac{u}{(1-u)}\frac{\delta}{(1-\delta)}\lambda n_2^{\delta}\,(N-n_1) \\
&\quad -\lambda n_2\left(\beta\frac{v}{1-v}\frac{n_1^{\delta}}{(1-\delta)} + (1-\beta)\,\frac{u}{1-u}n_2^{\delta}\right) \\
\dot{v} &= -\lambda n_1^{\delta} v \\
\dot{u} &= -\lambda n_2^{\delta} u
\end{aligned}$$

The Jacobian is shown in the appendix : *file jack.tex*
where $A = \frac{1-\alpha(1-\beta)}{(1-\delta)(1-\alpha)}\rho$, $B_t = \lambda\frac{\delta^2}{1-\delta}(N - n_{1t} - n_{2t})$, $C_t = \frac{\lambda}{1-\delta}(\beta\frac{v_t}{1-v_t}n_{1t}^{\delta} + (1-\beta)\frac{u_t}{1-u_t}n_{2t}^{\delta})$, $D_t = \frac{\lambda\delta}{1-\delta}(\beta\frac{v_t}{1-v_t}n_{1t}^{\delta-1} + (1-\beta)\frac{u_t}{1-u_t}n_{2t}^{\delta-1})$.

The hyperplane $\dot{n}_1 = 0$ defines: ($\Gamma = \frac{1-\alpha(1-\beta)}{\lambda(1-\alpha)}\rho$)

$$\begin{aligned}
\tilde{v}_1(n_1, n_2, u) &= \frac{\Gamma - (1-\beta)\, n_2^{\delta}\frac{u}{1-u}}{\Gamma - (1-\beta)\, n_2^{\delta}\frac{u}{1-u} + \beta n_1^{\delta-1}\left[n_1 + \delta(N - n_1 - n_2)\right]} \\
\dot{n}_1 &< 0\ \forall v > \tilde{v}_1,\ \dot{n}_1 > 0\ \forall v \in (0, \tilde{v}_1) \\
\tilde{u}_1(n_1, n_2, v) &= \frac{\Gamma - \beta\frac{v}{1-v}n_1^{\delta-1}\left[n_1 + \delta(N - n_1 - n_2)\right]}{\Gamma - \beta\frac{v}{1-v}n_1^{\delta-1}\left[n_1 + \delta(N - n_1 - n_2)\right] + (1-\beta)\, n_2^{\delta}} \\
\dot{n}_1 &< 0\ \forall u > \tilde{u}_1,\ \dot{n}_1 > 0\ \forall u \in (0, \tilde{u}_1)
\end{aligned}$$

while $\dot{n}_1 = 0$ for $n_1 = 0$ or n_1 et n_2 such that:

$$\begin{aligned}
H(n_1, n_2, v, u) &= \Gamma - (1-\beta)\, n_2^{\delta}\frac{u}{1-u} - \beta\frac{v}{1-v}n_1^{\delta-1}\left[n_1 + \delta(N - n_1 - n_2)\right] = 0 \\
\text{with } \frac{\partial H}{\partial n_1} &= \delta(1-\delta)\,\beta\frac{v}{1-v}n_1^{\delta-2}(N - n_1 - n_2) > 0 \\
\frac{\partial H}{\partial n_2} &= \delta\left[\beta\frac{v}{1-v}n_1^{\delta-1} - (1-\beta)\frac{u}{1-u}n_2^{\delta-1}\right] \gtreqless 0 \Leftrightarrow \frac{n_1}{n_2} \gtreqless \left[\frac{\beta}{1-\beta}\frac{v}{u}\frac{1-u}{1-v}\right]^{\frac{1}{1-\delta}} \\
\frac{\partial H}{\partial v} &= -\frac{\beta n_1^{\delta-1}}{(1-v)^2}\left[\delta(N - n_1 - n_2) + n_1\right] < 0 \\
\frac{\partial H}{\partial u} &= -\frac{(1-\beta)\, n_2^{\delta}}{(1-u)^2} < 0 \\
\text{hence } \frac{dn_1}{dn_2} &= \frac{n_1\left[\frac{1-\beta}{\beta}\frac{u}{v}\frac{1-v}{1-u}\left(\frac{n_1}{n_2}\right)^{1-\delta} - 1\right]}{(1-\delta)(N - n_1 - n_2)} \gtreqless 0 \Leftrightarrow \frac{n_1}{n_2} \lesseqgtr \left[\frac{\beta}{1-\beta}\frac{v}{u}\frac{1-u}{1-v}\right]^{\frac{1}{1-\delta}} \\
\frac{dn_1}{dv} &> 0 \text{ and } \frac{dn_1}{du} > 0\ ;\ \frac{dn_2}{dv} \lesseqgtr 0 \text{ and } \frac{dn_2}{du} \lesseqgtr 0 \Leftrightarrow \frac{n_1}{n_2} \gtreqless \left[\frac{\beta}{1-\beta}\frac{v}{u}\frac{1-u}{1-v}\right]^{\frac{1}{1-\delta}}
\end{aligned}$$

The hyperplane $\dot{n}_2 = 0$ defines the following functions:

$$\begin{aligned}
\tilde{v}_2(n_1, n_2, u) &= \frac{\Gamma - (1-\beta)\, n_2^{\delta-1}\frac{u}{1-u}\left[n_2 + \delta(N - n_1 - n_2)\right]}{\Gamma - (1-\beta)\, n_2^{\delta-1}\frac{u}{1-u}\left[n_2 + \delta(N - n_1 - n_2)\right] + \beta n_1^{\delta}} \\
\dot{n}_2 &< 0\ \forall v > \tilde{v}_2,\ \dot{n}_2 > 0\ \forall v \in (0, \tilde{v}_2) \\
\tilde{u}_2(n_1, n_2, v) &= \frac{\Gamma - \beta n_1^{\delta}\frac{v}{1-v}}{\Gamma - \beta n_1^{\delta}\frac{v}{1-v} + (1-\beta)\, n_2^{\delta-1}\left[n_2 + \delta(N - n_1 - n_2)\right]} \\
\dot{n}_2 &< 0\ \forall u \in (0, \tilde{u}_2),\ \dot{n}_2 > 0\ \forall u > \tilde{u}_2
\end{aligned}$$

while $\dot{n}_2 = 0$ for $n_2 = 0$ or n_1 et n_2 such that:

$$\begin{aligned}
G(n_1, n_2, v, u) &= \Gamma - (1-\beta)\, n_2^{\delta-1} \frac{u}{1-u}\left[n_2 + \delta(N - n_1 - n_2)\right] - \beta n_1^{\delta} \frac{v}{1-v} = 0 \\
\text{with } \frac{\partial G}{\partial n_1} &= \delta\left[(1-\beta)\frac{u}{1-u} n_2^{\delta-1} - \beta \frac{v}{1-v} n_1^{\delta-1}\right] \gtreqless 0 \Leftrightarrow \frac{n_1}{n_2} \gtreqless \left[\frac{\beta}{1-\beta}\frac{v}{u}\frac{1-u}{1-v}\right]^{\frac{1}{1-\delta}} \\
\frac{\partial G}{\partial n_2} &= \delta(1-\delta)(1-\beta)\frac{u}{1-u} n_2^{\delta-2}(N - n_1 - n_2) > 0 \\
\frac{\partial G}{\partial v} &= -\frac{\beta n_1^{\delta}}{(1-v)^2} < 0 \\
\frac{\partial G}{\partial u} &= -\frac{(1-\beta)\, n_2^{\delta-1}}{(1-u)^2}\left[n_2 + \delta(N - n_1 - n_2)\right] < 0 \\
\text{hence } \frac{dn_1}{dn_2} &= \frac{(1-\delta)(N - n_1 - n_2)}{n_2\left[\frac{\beta}{1-\beta}\frac{v}{u}\frac{1-u}{1-v}\left(\frac{n_1}{n_2}\right)^{\delta-1} - 1\right]} \lesseqgtr 0 \Leftrightarrow \frac{n_1}{n_2} \gtreqless \left[\frac{\beta}{1-\beta}\frac{v}{u}\frac{1-u}{1-v}\right]^{\frac{1}{1-\delta}} \\
\frac{dn_2}{dv} &> 0 \text{ and } \frac{dn_2}{du} > 0\ ;\ \frac{dn_1}{dv} > 0 \text{ and } \frac{dn_1}{du} > 0 \Leftrightarrow \frac{n_1}{n_2} \gtreqless \left[\frac{\beta}{1-\beta}\frac{v}{u}\frac{1-u}{1-v}\right]^{\frac{1}{1-\delta}}
\end{aligned}$$

5.2.3 Differentiating between control variables

Let us restate the problem and derive the dynamic system for the optimal solution, defining n as the aggregate amount of labor employed in R&D, and ϕ the share of these resources devoted to R&D for the non renewable resource. We drop the time subscript. The current value *Hamiltonian*:

$$H_c = \left[(N-n)(1-v)^{\beta}(1-u)^{1-\beta}\right]^{1-\alpha}\left(x^{\beta}s^{1-\beta}\right)^{\alpha} - cs - \mu x + \kappa\lambda\phi^{\delta}n^{\delta}v + \zeta\lambda(1-\phi)^{d} n^{\delta} u$$

whose first order conditions are:

$$(1-\alpha)\frac{Y}{(N-n)} = \delta\lambda n^{\delta-1}\left[\kappa\phi^{\delta}v + \zeta(1-\phi)^{\delta}u\right] \tag{42}$$

$$\phi = \frac{\left(\frac{\kappa v}{\zeta u}\right)^{\frac{1}{1-\delta}}}{1 + \left(\frac{\kappa v}{\zeta u}\right)^{\frac{1}{1-\delta}}} \tag{43}$$

$$\alpha\beta\frac{Y}{x} = \mu \tag{44}$$

$$\alpha(1-\beta)\frac{Y}{s} = c \tag{45}$$

Notice that (43) is equivalent to $\frac{\phi}{1-\phi} = \frac{n_1}{n_2} = \left(\frac{\kappa v}{\zeta u}\right)^{\frac{1}{1-\delta}}$, as it was found in the previous version. The Euler conditions are:

$$\begin{aligned}
\frac{\dot{\mu}}{\mu} &= \rho &(46)\\
\frac{\dot{\kappa}}{\kappa} &= \rho + \lambda\phi^{\delta}n^{\delta} - \beta\left(1-\alpha\right)\frac{Y}{\kappa\left(1-v\right)} &(47)\\
\frac{\dot{\zeta}}{\zeta} &= \rho + \lambda\left(1-\phi\right)^{\delta}n^{\delta} - \left(1-\beta\right)\left(1-\alpha\right)\frac{Y}{\zeta\left(1-u\right)} &(48)
\end{aligned}$$

The first technical procedure is designed to express the growth rate of the costate variables as function of the controls and states only. Substitute for $(1-\alpha)Y$ using (42) into (47) we have:

$$\frac{\dot{\kappa}}{\kappa} = \rho + \lambda\phi^{\delta}n^{\delta} - \frac{\beta\left(N-n\right)}{\kappa\left(1-v\right)}\delta\lambda n^{\delta-1}\left[\kappa\phi^{\delta}v + \zeta\left(1-\phi\right)^{\delta}u\right]$$

in which we substitute for $\zeta u = \kappa v\left[\left(1-\phi\right)/\phi\right]^{1-\delta}$ from (43) to get:

$$\begin{aligned}
\frac{\dot{\kappa}}{\kappa} &= \rho + \lambda\phi^{\delta}n^{\delta} - \frac{\beta\left(N-n\right)}{\kappa\left(1-v\right)}\delta\lambda n^{\delta-1}\left[\kappa\phi^{\delta}v + \kappa v\frac{\left(1-\phi\right)}{\phi^{1-\delta}}\right]\\
\text{''} &= \rho + \lambda\phi^{\delta}n^{\delta} - \frac{\beta\left(N-n\right)}{\kappa\left(1-v\right)}\delta\lambda n^{\delta-1}\kappa\phi^{\delta}v\left[1 + \frac{\left(1-\phi\right)}{\phi}\right]\\
\text{''} &= \rho + \lambda\phi^{\delta}n^{\delta} - \beta\delta\lambda n^{\delta-1}\phi^{\delta-1}\frac{v}{\left(1-v\right)}\left(N-n\right) &(49)
\end{aligned}$$

Applying the procedure to the growth rate of ζ we find:

$$\frac{\dot{\zeta}}{\zeta} = \rho + \lambda\left(1-\phi\right)^{\delta}n^{\delta} - \left(1-\beta\right)\delta\lambda n^{\delta-1}\left(1-\phi\right)^{\delta-1}\frac{u}{\left(1-u\right)}\left(N-n\right) \qquad (50)$$

Next we want to derive an expression describing the optimal growth rate of the share of R&D allocated to the non renewable resource (it measures the "direction" of R&D). Take logs and differentiate (43) written as $\left(\frac{\phi}{1-\phi}\right)^{1-\delta} = \frac{\kappa v}{\zeta u}$ and use (49) and (50):[6]

$$\begin{aligned}
\left(\frac{1-\delta}{1-\phi}\right)\frac{\dot{\phi}}{\phi} &= \frac{\dot{\kappa}}{\kappa} + \frac{\dot{v}}{v} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{u}}{u}\\
\text{''} &= \rho + \lambda\phi^{\delta}n^{\delta} - \beta\delta\lambda n^{\delta-1}\phi^{\delta-1}\frac{v}{\left(1-v\right)}\left(N-n\right) - \lambda\phi^{\delta}n^{\delta}\\
&\quad -\rho - \lambda\left(1-\phi\right)^{\delta}n^{\delta} + \left(1-\beta\right)\delta\lambda n^{\delta-1}\left(1-\phi\right)^{\delta-1}\frac{u}{\left(1-u\right)}\left(N-n\right) + \lambda\left(1-\phi\right)^{\delta}n^{\delta}\\
\text{''} &= \delta\lambda n^{\delta-1}\left(N-n\right)\left[\left(1-\beta\right)\left(1-\phi\right)^{\delta-1}\frac{u}{\left(1-u\right)} - \beta\phi^{\delta-1}\frac{v}{\left(1-v\right)}\right]
\end{aligned}$$

[6] Consider that $d\log(\phi/\left(1-\phi\right))/dt = d\log\phi/dt - d\log\left(1-\phi\right)/dt = \dot{\phi}/\phi + \dot{\phi}/\left(1-\phi\right) = \dot{\phi}/\phi[1+\phi/(1-\phi)] = \dot{\phi}/[\phi(1-\phi)]$.

and the result follows:

$$\frac{\dot{\phi}}{\phi}=\frac{\delta}{1-\delta}\lambda\left(1-\phi\right)n^{\delta-1}\left(N-n\right)\left[\left(1-\beta\right)\left(1-\phi\right)^{\delta-1}\frac{u}{\left(1-u\right)}-\beta\phi^{\delta-1}\frac{v}{\left(1-v\right)}\right] \tag{51}$$

To derive the growth rate of n first log differentiate the production function as written in H:

$$\frac{\dot{Y}}{Y}=\left(1-\alpha\right)\left[-\frac{\dot{n}}{\left(N-n\right)}-\beta\frac{\dot{v}}{1-v}-\left(1-\beta\right)\frac{\dot{u}}{1-u}\right]+\alpha\left[\beta\frac{\dot{x}}{x}+\left(1-\beta\right)\frac{\dot{s}}{s}\right]$$

>From (44), (45) and (46) we have $\dot{s}/s=\dot{Y}/Y$ and $\dot{x}/x=\dot{Y}/Y-\rho$, thus $\beta\frac{\dot{x}}{x}+\left(1-\beta\right)\frac{\dot{s}}{s}=\dot{Y}/Y-\beta\rho$. The expression above can therefore be rearranged as follows:and $\frac{\dot{s}}{s}-\rho$. Taking these equalities into account together with the result from ().substituting in the previous expression:

$$\frac{\dot{Y}}{Y}-\frac{\dot{n}}{\left(N-n\right)}=-\beta\frac{\dot{v}}{1-v}-\left(1-\beta\right)\frac{\dot{u}}{1-u}-\frac{\alpha\beta}{\left(1-\alpha\right)}\rho$$

notice that if $\dot{v},\dot{u}\rightarrow 0$ and $\dot{n}\rightarrow 0$, then $\dot{Y}/Y\rightarrow-\frac{\alpha\beta}{1-\alpha}\rho\leq 0$, with equality only if output is inelastic w.r.t. fossil fuel inputs ($\beta=0$). Next, substituting for $\zeta u=\kappa v\left[\left(1-\phi\right)/\phi\right]^{1-\delta}$ from (43) into (42), we obtain $\left(1-\alpha\right)Y/\left(N-n\right)=\kappa\delta\lambda n^{\delta-1}\phi^{\delta-1}v$. Taking logs and differentiating:

$$\frac{\dot{Y}}{Y}-\frac{\dot{n}}{\left(N-n\right)}=\frac{\dot{\kappa}}{\kappa}+\frac{\dot{v}}{v}-\left(1-\delta\right)\frac{\dot{n}}{n}-\left(1-\delta\right)\frac{\dot{\phi}}{\phi}$$

equating the last two expressions:

$$\left(1-\delta\right)\frac{\dot{n}}{n}=\frac{\alpha\beta}{\left(1-\alpha\right)}\rho+\beta\frac{\dot{v}}{1-v}+\left(1-\beta\right)\frac{\dot{u}}{1-u}+\frac{\dot{\kappa}}{\kappa}+\frac{\dot{v}}{v}-\left(1-\delta\right)\frac{\dot{\phi}}{\phi}$$

Now we can substitute using (49) and (51) :

$$\begin{aligned}
(1-\delta)\frac{\dot{n}}{n} &= \frac{\alpha\beta}{(1-\alpha)}\rho-\beta\frac{v}{1-v}\lambda\phi^{\delta}n^{\delta}-(1-\beta)\frac{u}{1-u}\lambda(1-\phi)^{\delta}n^{\delta} \\
&+\rho+\lambda\phi^{\delta}n^{\delta}-\beta\delta\lambda n^{\delta-1}\phi^{\delta-1}\frac{v}{(1-v)}(N-n)-\lambda\phi^{\delta}n^{\delta} \\
&-(1-\delta)\frac{\delta}{1-\delta}\lambda(1-\phi)n^{\delta-1}(N-n)\left[(1-\beta)(1-\phi)^{\delta-1}\frac{u}{(1-u)}-\beta\phi^{\delta-1}\frac{v}{(1-v)}\right] \\
" &= \frac{1-\alpha(1-\beta)}{(1-\alpha)}\rho-\delta\lambda n^{\delta-1}(N-n)\left\{\beta\phi^{\delta-1}\frac{v}{(1-v)}-\beta\phi^{\delta-1}(1-\phi)\frac{v}{(1-v)}\right. \\
&\left.+(1-\beta)(1-\phi)^{\delta}\frac{u}{(1-u)}\right\}-\beta\frac{v}{1-v}\lambda\phi^{\delta}n^{\delta}-(1-\beta)\frac{u}{1-u}\lambda(1-\phi)^{\delta}n^{\delta} \\
" &= \frac{1-\alpha(1-\beta)}{(1-\alpha)}\rho-\delta\lambda n^{\delta-1}(N-n)\left[\beta\phi^{\delta}\frac{v}{(1-v)}+(1-\beta)(1-\phi)^{\delta}\frac{u}{(1-u)}\right] \\
&-\lambda n^{\delta}\left[\beta\phi^{\delta}\frac{v}{1-v}+(1-\beta)(1-\phi)^{\delta}\frac{u}{1-u}\right] \\
" &= \frac{1-\alpha(1-\beta)}{(1-\alpha)}\rho-\lambda n^{\delta}\left(1+\delta\frac{N-n}{n}\right)\left[\beta\phi^{\delta}\frac{v}{1-v}+(1-\beta)(1-\phi)^{\delta}\frac{u}{1-u}\right]
\end{aligned}$$

therefore:

$$\frac{\dot{n}}{n}=\frac{1-\alpha(1-\beta)}{(1-\alpha)(1-\delta)}\rho-\lambda n^{\delta-1}\left(\frac{\delta}{(1-\delta)}N+n\right)\left[\beta\phi^{\delta}\frac{v}{1-v}+(1-\beta)(1-\phi)^{\delta}\frac{u}{1-u}\right] \tag{52}$$

To summarize our dynamic system is:

$$\begin{aligned}
\frac{\dot{n}}{n} &= \frac{1-\alpha(1-\beta)}{(1-\alpha)(1-\delta)}\rho-\lambda n^{\delta-1}\left(\frac{\delta}{(1-\delta)}N+n\right)\left[\beta\phi^{\delta}\frac{v}{1-v}+(1-\beta)(1-\phi)^{\delta}\frac{u}{1-u}\right] \\
\frac{\dot{\phi}}{\phi} &= \frac{\delta}{1-\delta}\lambda(1-\phi)n^{\delta-1}(N-n)\left[(1-\beta)(1-\phi)^{\delta-1}\frac{u}{(1-u)}-\beta\phi^{\delta-1}\frac{v}{(1-v)}\right] \\
\frac{\dot{v}}{v} &= \lambda\phi^{\delta}n^{\delta} \\
\frac{\dot{u}}{u} &= \lambda(1-\phi)^{\delta}n^{\delta}
\end{aligned}$$

Remark 1 *The case $\dot{\phi}=0$ and $\dot{n}=0$ is not optimal.*

Proof. From (52) $\dot{n}/n=0$ if:

$$\frac{\phi}{1-\phi}=\left(\frac{1-\beta}{\beta}\frac{u}{1-u}\frac{1-v}{v}\right)^{\frac{1}{\delta}}$$

while (51) implies that $\dot{\phi}/\phi=0$ if:

$$\frac{\phi}{1-\phi}=\left(\frac{1-\beta}{\beta}\frac{u}{1-u}\frac{1-v}{v}\right)^{\frac{1}{\delta-1}}$$

Hence $\dot{\phi} = 0$ and $\dot{n} = 0$ if and only if $\phi = .5$, implying:

$$\frac{v}{1-v} = \frac{1-\beta}{\beta}\frac{u}{1-u}$$

and this arbitrage condition should hold over time. taking logs and differentiating, the condition implies:

$$\frac{\lambda\phi^{\delta}n^{\delta}}{1-v} = \frac{\dot{v}}{v}\frac{1}{1-v} = \frac{\dot{u}}{u}\frac{1}{1-u} = \frac{\lambda(1-\phi)^{\delta}n^{\delta}}{1-u}$$

that is:

$$\frac{\phi}{1-\phi} = \left(\frac{1-v}{1-u}\right)^{\frac{1}{\delta}}$$

which together with $\phi = .5$, gives $v = u$ and $\beta = .5$. Unfortunately β is an exogenous parameter included in the interval $(0, 1)$ and its value cannot be restricted. ■

5.3 Appendix 3: Decentralized Economy, No Substitute

5.3.1 No Substitute 1[7]

The aim of this note is to check what are the implications of assuming:

1. that the economy depends upon a non renewable resource, x, available in a given stock of size X;
2. that technological progress takes the following particular form: energy efficiency is bounded below unity (thermodynamics principles) and can approach this upper bound asymptotically if there is continuous R&D activity.

The model is described by the following:
The final sector's production function:

$$Y_t = L_t^{1-\alpha}\int_0^1 (1-v_{jt})\, e_{jt}^{\alpha} dj \tag{53}$$

with $\alpha \in (0, 1)$, L denotes labor input, and e_j is the "energy good" sold by the local monopolist of technology v_j.
The monopolist produces e from the non renewable resource according to, $\forall j \in [0, 1]$:

$$e_{jt} = x_{jt} \tag{54}$$

[7] Notes on the Fossil Fuel Bias project (first draft 12/01/02 ; this one 25/2/02 (not so sure)).

The law of motion of the resource stock:

$$\dot{X}_t = -x_t = -\int_0^1 x_{jt} dj \qquad \forall t \; X, \; x \geq 0 \tag{55}$$

The law of motion of the leading edge energy efficiency (defined by $v^m = \min\{v_j\} - \varepsilon$) resulting from R&D:

$$\frac{\dot{v}_t^m}{v_t^m} = -\sigma\phi\left(n_t\right) \tag{56}$$

where n is R&D employment, $\phi' > 0$, $\phi'' \leq 0$, and the aggregate flow of innovations is:

$$\phi\left(n_t\right) = \int_0^1 \phi\left(n_{jt}\right) dj$$

The labor market clears:

$$L + n = N \tag{57}$$

N, X_0, v_0^m, α, σ, λ are exogenous parameters. Define the average efficiency index:

$$v_t = \int_0^1 v_{jt} dj \tag{58}$$

Final sector. Perfect competition. Taking as given the wage, w, and the price of the "energy goods", p_j $\forall j$, the fictitious final sector firm problem is:

$$\max_{L_t, \{e_{jt}\}_{j=0}^1} L_t^{1-\alpha} \int_0^1 \left(1 - v_{jt}\right) e_{jt}^{\alpha} dj - w_t L_t - \int_0^1 p_{jt} e_{jt} dj$$

>From which one obtains the demand function for labor from the final sector:

$$w_t = \left(1-\alpha\right) L_t^{-\alpha} \int_0^1 \left(1 - v_{jt}\right) e_{jt}^{\alpha} dj \quad \text{or} \quad L^d = \left(\frac{1-\alpha}{w_t}\right)^{\frac{1}{\alpha}} \left(\int_0^1 \left(1 - v_{jt}\right) e_{jt}^{\alpha} dj\right)^{\frac{1}{\alpha}} \tag{59}$$

and the demand for "energy good" $\forall j \in [0, 1]$:

$$p_{jt} = \alpha\left(1 - v_{jt}\right)\left(\frac{L_t}{e_{jt}}\right)^{1-\alpha} \qquad \text{or} \quad e_{jt}^d = \left(\frac{\alpha\left(1 - v_{jt}\right)}{p_{jt}}\right)^{\frac{1}{1-\alpha}} L_t \tag{60}$$

Local monopolist. Consider a monopolist in sector j endowed of the best technology within the sector, v_j at a given date. Under the technology of production (54), the marginal and unit cost of the monopolist is the unit price of the non renewable resource, which I denote by p_x. The monopolist problem is:

$$\max_{e_{jt}} \pi_{jt} \Leftrightarrow \max_{e_{jt}} (p_{jt} - p_{xt}) e_{jt} \qquad \text{s.t. } (60)$$

which gives

$$p_{xt} = \alpha^2 (1 - v_j) \left(\frac{L_t}{e_{jt}} \right)^{1-\alpha}$$

and using again (60):

$$\begin{aligned} \hat{p}_{jt} &= \frac{p_{xt}}{\alpha} \\ \hat{e}_{jt} &= \left(\frac{\alpha^2 (1 - v_j)}{p_{xt}} \right)^{\frac{1}{1-\alpha}} L_t \qquad (61) \\ \hat{\pi}_{jt} &= (1-\alpha) p_{jt} e_{jt} = \frac{1-\alpha}{\alpha} p_{xt} e_{jt} \qquad (62) \\ " &= \frac{1-\alpha}{\alpha} \left[\alpha^2 (1 - v_j) \right]^{\frac{1}{1-\alpha}} L_t p_{xt}^{\frac{-\alpha}{1-\alpha}} \end{aligned}$$

R&D. The R&D input is labor, and the marginal cost is the wage, w.

R&D is a stochastic activity whose output is governed by a Poisson process (no memory). Employing a marginal researcher on product line j, firm i increases by $\phi' (n_{ij})$ the instantaneous probability of innovating. Let me assume $\phi' (n_{ij}) = \phi' (n_j)\ \forall i$. An innovator obtains the patent to produce the good with the leading edge efficiency at the date of innovation. Therefore the value,V_t, of an innovation arrived at date t is the expected present value of the stream of profits. Since the innovator obtains the same technology regardless to the sector in which it innovates, R&D employment is uniform across sectors. Given that there is a unit mass of sectors: $\phi' (n_j) = \phi' (n)$ and $\phi (n_j) = \phi (n)$. Using (62):

$$\begin{aligned} V_t &= E_t \left(\int_t^{\infty} e^{-\int_t^s r_u du} e^{-\int_t^s \phi(n_u) du} \frac{1-\alpha}{\alpha} \left[\alpha^2 (1 - v_t^m) \right]^{\frac{1}{1-\alpha}} L_s p_{xs}^{\frac{-\alpha}{1-\alpha}} ds \right) \\ " &= \frac{1-\alpha}{\alpha} \left[\alpha^2 (1 - v_t^m) \right]^{\frac{1}{1-\alpha}} E_t \left(\int_t^{\infty} e^{-\int_t^s r_u du} e^{-\int_t^s \phi(n_u) du} L_s p_{xs}^{\frac{-\alpha}{1-\alpha}} ds \right) \qquad (63) \end{aligned}$$

The *R&D arbitrage condition* equates the marginal cost to the expected marginal reward, that is, if $n_t > 0$:

$$w_t = \phi' (n_t) V_t \qquad (64)$$

Resource Market. Using (61) I can compute the instantaneous aggregate demand for the resource:

$$\begin{aligned} x_t^d &= \int_0^1 \hat{e}_{jt} dj = \left(\frac{\alpha^2}{p_{xt}}\right)^{\frac{1}{1-\alpha}} L_t \int_0^1 (1-v_{jt})^{\frac{1}{1-\alpha}} dj \\ " &= \left(\frac{\alpha^2}{p_{xt}}\right)^{\frac{1}{1-\alpha}} L_t \omega_t \end{aligned}$$

where I have defined the *efficiency index*:

$$\omega_t = \int_0^1 (1-v_{jt})^{\frac{1}{1-\alpha}} dj < 1 - v_t$$

Assume perfect competition on the market. Resource suppliers take the price, p_x, as given. Let the unit extraction cost be constant and equal to b. Define the *resource rent* as:

$$q_t = p_{xt} - b$$

The *Hotelling rule* can be derived with a simple arbitrage argument: delaying extraction does not modify the extraction cost, hence the marginal return on holding the non-extracted resource as an asset consists of its capital gain, which is the appreciation of the rent. This implicit return must equal the return on the riskless asset, r:

$$\frac{\dot{q}_t}{q_t} = r_t \tag{65}$$

Define by q_0 the initial rent, then the *price of the resource* is

$$p_{xt} = b + q_0 e^{\int_0^t r_u du} \tag{66}$$

Use this in the aggregate demand function to impose the *resource market-clearing condition*:

$$x_t = \alpha^{\frac{2}{1-\alpha}} L_t \omega_t \left(b + q_0 e^{\int_0^t r_u du}\right)^{\frac{-1}{1-\alpha}} \tag{67}$$

Condition (67) must hold at any instant. However, the extraction path must be compatible with the limited availability of the resource. This constraint allows us to pin down the unique value of the initial resource rent, q_0. The *exhaustion condition* is:

$$X_0 = \int_0^\infty x_t dt = \alpha^{\frac{2}{1-\alpha}} \int_0^\infty L_t \omega_t \left(b + q_0 e^{\int_t^s r_u du}\right)^{-\frac{1}{1-\alpha}} dt \tag{68}$$

There is no balanced path with constant R&D employment. Suppose that there exists a (balanced) path such that:

1. $r_t = r \; \forall t$, constant interest rate (small open economy, linear instantaneous utility) ;

2. $n_t = n \; \forall t$, the solution is characterized by constant R&D employment.

2 is satisfied only if the R&D arbitrage condition (64) holds for a constant level of n at all dates.

Let me reduce the model to a system of two equation in two unknowns, n and q_0. First, using (61) in the labor demand function from the final sector (59), I have:

$$w_t = (1-\alpha)\,\alpha^{\frac{2\alpha}{1-\alpha}} p_{xt}^{\frac{-\alpha}{1-\alpha}} \omega_t$$

Let me also simplify by assuming $\phi(n) = \lambda n$. Then (63) can be written (dropping the expectation operator and taking into account):

$$V_t = \frac{1-\alpha}{\alpha}\alpha^{\frac{2}{1-\alpha}} \left(1 - v_t^m\right)^{\frac{1}{1-\alpha}} (N-n) \int_t^{+\infty} e^{-(r+\lambda n)(s-t)} p_{xs}^{\frac{-\alpha}{1-\alpha}} ds$$

plugging these two equations in (64) and simplifying, we get:

$$\omega_t = \lambda\alpha \left(1 - v_t^m\right)^{\frac{1}{1-\alpha}} (N-n) \int_t^{+\infty} e^{-(r+\lambda n)(s-t)} \left(\frac{p_{xt}}{p_{xs}}\right)^{\frac{\alpha}{1-\alpha}} ds$$

Now I turn to two very special terms.

First, using the equilibrium price of the resource (66):

$$\begin{aligned}
\int_t^{+\infty} e^{-(r+\lambda n)(s-t)} \left(\frac{p_{xt}}{p_{xs}}\right)^{\frac{\alpha}{1-\alpha}} ds &= \int_t^{+\infty} e^{-(r+\lambda n)(s-t)} \left(\frac{b+q_0 e^{rt}}{b+q_0 e^{rs}}\right)^{\frac{\alpha}{1-\alpha}} ds \\
\text{''} &= \int_0^{+\infty} e^{-(r+\lambda n)u} \left(\frac{b+q_0 e^{rt}}{b+q_0 e^{r(t+u)}}\right)^{\frac{\alpha}{1-\alpha}} du
\end{aligned}$$

The second term in the integrand converges to zero and faster the greater is t.

Second, we can study the behavior of the efficiency indexes. I proceed to evaluate it by adopting as the space of integration that of the age $s \in [0,t]$. Let $h(s,t)$ denote the mass of goods of age s still on the market at date t. We have that $\lim_{t\to\infty} h(s,t) = \lambda n e^{-\lambda n s}$, because a proportion $e^{-\lambda n s}$ of goods of age s has survived out of the initial mass λn. However, initially the distribution could be different, and its choice is actually arbitrary. Assuming that at date $t=0$ all goods share the same technology v_0^m (i.e. $\forall s \in (-\infty, 0)\; h(s,0) = 0$ and $h(0,0) = 1$) we have (see Appendix 5.4):

$$h(s,t) = \frac{\lambda n e^{-\lambda n s}}{1 - e^{-\lambda n t}}$$

Using $v_{s,t} = v_{t-s}^m = v_0^m e^{-\sigma\lambda n(t-s)}$, we can write:

$$
\begin{aligned}
\frac{\omega_t}{(1-v_t^m)^{\frac{1}{1-\alpha}}} &= \int_0^1 \left(\frac{1-v_{jt}}{1-v_t^m}\right)^{\frac{1}{1-\alpha}} dj = \int_0^t \left(\frac{1-v_s}{1-v_t^m}\right)^{\frac{1}{1-\alpha}} h(s,t)\,ds \\
\text{''} &= \int_0^t \left(\frac{1-v_0^m e^{\sigma\lambda n(s-t)}}{1-v_0^m e^{-\sigma\lambda nt}}\right)^{\frac{1}{1-\alpha}} h(s,t)\,ds \\
\text{''} &= \int_0^t \left(\frac{1-v_0^m e^{\sigma\lambda n(s-t)}}{1-v_0^m e^{-\sigma\lambda nt}}\right)^{\frac{1}{1-\alpha}} \frac{\lambda n e^{-\lambda ns}}{1-e^{-\lambda nt}} ds \in (0,1)
\end{aligned}
$$

This object measures the technological distance of the leading edge sector from the average sector. It converges to unity asymptotically with t. This non stationary behavior explains why balanced paths do not exist.

Merging the last three equations, I write the (instantaneous) **General Equilibrium condition**, that should give the constant R&D employment along the balanced path as function of q_0, $\forall t$:

$$
\int_0^t \left(\frac{1-v_0^m e^{\sigma\lambda n(s-t)}}{1-v_0^m e^{-\sigma\lambda nt}}\right)^{\frac{1}{1-\alpha}} \frac{\lambda n e^{-\lambda ns}}{1-e^{-\lambda nt}} ds = \lambda\alpha(N-n)\int_0^\infty e^{-(r+\lambda n)s}\left(\frac{b+q_0 e^{rt}}{b+q_0 e^{r(t+s)}}\right)^{\frac{\alpha}{1-\alpha}} ds \tag{GE}
$$

Finally, I rewrite condition (68) using ω above and (57), to get the **Full Exhaustion condition**:

$$
X_0 = \alpha^{\frac{2}{1-\alpha}}(N-n)\int_0^\infty (b+q_0e^{ru})^{\frac{-1}{1-\alpha}} \left(\int_0^u \left[1-v_0^m e^{\sigma\lambda n(s-u)}\right]^{\frac{1}{1-\alpha}} \frac{\lambda n e^{-\lambda ns}}{1-e^{-\lambda nu}} ds\right) du \tag{FE}
$$

and gives the initial resource rent, q_0, as function of n.

Remark 2 *The problem is non stationary because of the form of technological progress that was assumed. This is clear because (GE) is non stationary even if $b = q_0 = 0$, that is if the resource is free and renewable.*

5.3.2 No Substitute 2: Simplified Version (*P&K*)

In this section the previous model is simplified assuming that the monopolist produces e from the non renewable resource according to, $\forall j \in [0,1]$:

$$
e_{jt} = \frac{x_{jt}}{(1-v_{jt})} \tag{69}
$$

implying that a better technology is also more resource intensive. In this case the unit cost for the monopolist producing with technology v_j is $(1-v_j)\,p_{xt}$.

The monopolist problem is modified to:

$$\max_{e_{jt}} \pi_{jt} \;\Leftrightarrow\; \max_{e_{jt}} \left(p_{jt} - (1 - v_j)\, p_{xt}\right) e_{jt} \qquad \text{s.t. } (60)$$

which gives

$$p_{xt} = \alpha^2 \left(\frac{L_t}{e_{jt}}\right)^{1-\alpha}$$

and using again (60):

$$\begin{aligned}
\hat{p}_{jt} &= (1 - v_j)\, \frac{p_{xt}}{\alpha} \\
\hat{e}_{jt} &= \left(\frac{\alpha^2}{p_{xt}}\right)^{\frac{1}{1-\alpha}} L_t \\
\hat{\pi}_{jt} &= \frac{1-\alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} (1 - v_j)\, L_t p_{xt}^{\frac{-\alpha}{1-\alpha}}
\end{aligned} \tag{70}$$

As consequence the value of a patent for producing with technology v_t^m is proportional to the efficiency index of the technology:

$$\begin{aligned}
V_t &= E_t \left(\int_t^\infty e^{-\int_t^s r_u du} e^{-\int_t^s \phi(n_u) du} \frac{1-\alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} (1 - v_t^m)\, L_s p_{xs}^{\frac{-\alpha}{1-\alpha}} ds \right) \\
" &= \frac{1-\alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} (1 - v_t^m)\, E_t \left(\int_t^\infty e^{-\int_t^s r_u du} e^{-\int_t^s \phi(n_u) du} L_s p_{xs}^{\frac{-\alpha}{1-\alpha}} ds \right)
\end{aligned}$$

Concerning the resource market we have:

$$\begin{aligned}
x_t^d &= \int_0^1 (1 - v_{jt})\, \hat{e}_{jt} dj = \left(\frac{\alpha^2}{p_{xt}}\right)^{\frac{1}{1-\alpha}} L_t \int_0^1 (1 - v_{jt})\, dj \\
" &= \left(\frac{\alpha^2}{p_{xt}}\right)^{\frac{1}{1-\alpha}} L_t \tilde{\omega}_t
\end{aligned}$$

Notice that now the crucial variable in aggregate demand is simply the *average efficiency index*:

$$\tilde{\omega}_t = \int_0^1 (1 - v_{jt})\, dj = 1 - v_t$$

which is relatively simple to compute, knowing the cross sectoral distribution of technologies. Now the *resource market-clearing condition* is*:*

$$x_t = \alpha^{\frac{2}{1-\alpha}} L_t \tilde{\omega}_t \left(b + q_0 e^{\int_t^s r_u du} \right)^{\frac{-1}{1-\alpha}} \tag{71}$$

Condition (67) must hold at any instant. However, the extraction path must be compatible with the limited availability of the resource. This constraint allows us to pin down the unique value of the initial resource rent, q_0. The *exhaustion condition* is:

$$X_0 = \int_0^\infty x_t dt = \alpha^{\frac{2}{1-\alpha}} \int_0^\infty L_t \tilde{\omega}_t \left(b + q_0 e^{\int_t^s r_u du} \right)^{\frac{-1}{1-\alpha}} dt \tag{72}$$

There is no balanced path with constant R&D employment. Suppose that there exists a (balanced) path such that:

1. $r_t = r \ \forall t$, constant interest rate (small open economy, linear instantaneous utility) ;
2. $n_t = n \ \forall t$, the solution is characterized by constant R&D employment.

2 is satisfied only if the R&D arbitrage condition (64) holds for a constant level of n at all dates.

Let me reduce the model to a system of two equation in two unknowns, n and q_0. First, using (70) in the labor demand function from the final sector (59), I have:

$$w_t = (1-\alpha)\,\alpha^{\frac{2\alpha}{1-\alpha}} p_{xt}^{\frac{-\alpha}{1-\alpha}} \tilde{\omega}_t$$

Let me also simplify by assuming $\phi(n) = \lambda n$. Then (63) can be written (dropping the expectation operator and taking into account):

$$V_t = \frac{1-\alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} (1 - v_t^m)(N-n) \int_t^\infty e^{-(r+\lambda n)(s-t)} p_{xs}^{\frac{-\alpha}{1-\alpha}} ds$$

plugging these two equations in (64) and simplifying, we get:

$$\tilde{\omega}_t = \lambda \alpha (1 - v_t^m)(N-n) \int_t^\infty e^{-(r+\lambda n)(s-t)} \left(\frac{p_{xt}}{p_{xs}} \right)^{\frac{\alpha}{1-\alpha}} ds$$

Using $h(s,t) = \lambda n e^{-\lambda n s}\left(1-e^{-\lambda n t}\right)$ and $v_{st} = v^m_{t-s} = v^m_0 e^{-\sigma\lambda n(t-s)}$, we have:

$$\begin{aligned}
\tilde{\omega}_t &= \int_0^1 (1-v_{jt})\, dj = \int_0^t (1-v_{st})\, h(s,t)\, ds \\
" &= \int_0^t \left(1 - v^m_t e^{\sigma\lambda n s}\right) h(s,t)\, ds \\
" &= \int_0^t \left(1 - v^m_t e^{\sigma\lambda n s}\right) \frac{\lambda n e^{-\lambda n s}}{\left(1-e^{-\lambda n t}\right)} ds \\
" &= \frac{\lambda n}{\left(1-e^{-\lambda n t}\right)} \left[\frac{1-e^{-\lambda n t}}{\lambda n} - \frac{v^m_t\left(1-e^{-(1-\sigma)\lambda n t}\right)}{(1-\sigma)\lambda n}\right] \\
" &= 1 - \frac{v^m_0 e^{-\sigma\lambda n t}}{(1-\sigma)} \frac{\left(1-e^{-(1-\sigma)\lambda n t}\right)}{\left(1-e^{-\lambda n t}\right)} \in (0,1) \qquad \text{if } \ \sigma < 1
\end{aligned}$$

that is:

$$1 - v_t = 1 - v^m_t \frac{\left(1-e^{-(1-\sigma)\lambda n t}\right)}{(1-\sigma)\left(1-e^{-\lambda n t}\right)}$$

This object measures the average efficiency and:

$$\lim_{t\to\infty} \tilde{\omega}_t = 1 - \frac{v^m_t}{1-\sigma} < 1 - v^m_t \quad \text{if } \ \sigma < 1$$

This finding implies that the technological distance of the leading edge sector from the average sector falls over time. The picture below depicts v_t/v^m_t.

The **General Equilibrium condition** is now $\forall t$:

$$1 - \frac{v^m_0 e^{-\sigma\lambda n t}}{(1-\sigma)} \frac{\left(1-e^{-(1-\sigma)\lambda n t}\right)}{\left(1-e^{-\lambda n t}\right)} = \left(1 - v^m_0 e^{-\sigma\lambda n t}\right)\lambda\alpha(N-n)\int_0^\infty e^{-(r+\lambda n)s}\left(\frac{b+q_0 e^{rt}}{b+q_0 e^{r(t+s)}}\right)^{\frac{\alpha}{1-\alpha}} ds \tag{GE}$$

and the **Full Exhaustion condition**:

$$X_0 = \alpha^{\frac{2}{1-\alpha}}(N-n)\int_0^\infty \left(b+q_0 e^{ru}\right)^{\frac{-1}{1-\alpha}}\left(1 - \frac{v^m_0 e^{-\sigma\lambda n u}}{(1-\sigma)}\frac{\left(1-e^{-(1-\sigma)\lambda n u}\right)}{\left(1-e^{-\lambda n u}\right)}\right) du \tag{FE}$$

and gives the initial resource rent, q_0, as function of n.

5.4 Appendix 4: Distribution of vintages across sectors

Recall the definition of $h(s,t)$: the mass of goods that at date t have age s, implying that they are characterized by a technology that was the leading edge one at date $t-s$. Assume that $\forall s \in (-\infty, 0)$ $h(s,0) = 0$ and $h(0,0) = 1$. In other words, suppose that at date $t = 0$ all goods share the same technology v_0^m. In this appendix, I derive heuristically the distribution function:

$$h(s,t) = \frac{\lambda n e^{-\lambda n s}}{1 - e^{-\lambda n t}}$$

Let me construct the table below:

date	**mass**	**technology**
0	$h(0,0) = 1$	v_0^m
$d\tau$	$h(d\tau, d\tau) = e^{-\lambda n d\tau} h(0,0) = e^{-\lambda n d\tau}$	v_0^m
	$h(0, d\tau) = 1 - \int_{d\tau}^{d\tau} h(s, d\tau)\, ds = 1 - e^{-\lambda n d\tau}$	$v_{d\tau}^m$
$2d\tau$	$h(2d\tau, 2d\tau) = e^{-\lambda n d\tau} h(d\tau, d\tau) = e^{-\lambda n 2 d\tau}$	v_0^m
	$h(d\tau, 2d\tau) = e^{-\lambda n d\tau} h(0, d\tau) = e^{-\lambda n d\tau} - e^{-\lambda n 2 d\tau}$	$v_{d\tau}^m$
	$h(0, 2d\tau) = 1 - \int_{d\tau}^{2d\tau} h(s, d\tau)\, ds = 1 - e^{-\lambda n d\tau}$	$v_{2d\tau}^m$
$3d\tau$	$h(3d\tau, 3d\tau) = e^{-\lambda n d\tau} h(2d\tau, 2d\tau) = e^{-\lambda n 3 d\tau}$	v_0^m
	$h(2d\tau, 3d\tau) = e^{-\lambda n d\tau} h(d\tau, 2d\tau) = e^{-\lambda n 2 d\tau} - e^{-\lambda n 3 d\tau}$	$v_{d\tau}^m$
	$h(d\tau, 3d\tau) = e^{-\lambda n d\tau} h(0, 2d\tau) = e^{-\lambda n d\tau} - e^{-\lambda n 2 d\tau}$	$v_{2d\tau}^m$
	$h(0, 3d\tau) = 1 - \int_{d\tau}^{3d\tau} h(s, d\tau)\, ds = 1 - e^{-\lambda n d\tau}$	$v_{3d\tau}^m$

Hence the distribution seems to behave according to $e^{-\lambda n s}\left(1 - e^{-\lambda n d\tau}\right)$ which tends to $e^{-\lambda n s}$ as $d\tau \to 0$. Recalling that there is a unit mass of sectors and using the fact that a flow λn of innovations enter the economy at each instant, the distribution should be described by the following law, if the memory of the initial distribution $h(0,0)$ had faded away: $\int_0^\infty \lambda n e^{-\lambda n s} ds = 1$. However in this model the initial distribution cannot be ruled out. In fact, the efficiency index $1 - v_{s,t}$ of old-enough goods turns negative if we let $s \to \infty$.[8] Hence if we restrict the domain of the distribution to $s \in [0,t]$ we have:

$$\begin{aligned}
\int_0^t h(s,t)\, ds &= \int_0^t \frac{\lambda n e^{-\lambda n s}}{1 - e^{-\lambda n t}} ds = \frac{\lambda n}{1 - e^{-\lambda n t}} \int_0^t e^{-\lambda n s} ds \\
\text{''} &= \frac{\lambda n}{1 - e^{-\lambda n t}} \left| \frac{-e^{-\lambda n s}}{\lambda n} \right|_0^t = \frac{\lambda n}{1 - e^{-\lambda n t}} \frac{-e^{-\lambda n t} + 1}{\lambda n} = 1
\end{aligned}$$

Finally, the property $\lim_{t \to \infty} h(s,t) = \lambda n e^{-\lambda n s}$ is clearly verified.

6 References

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[8] We have $1 - v_{s,t} = 1 - v_0^m e^{-\sigma \lambda n (t-s)} < 0$ for $\forall s > \bar{s} = t - \ln\left(v_0^m\right) / \sigma \lambda n$.

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